

# Global well - posedness and scattering for the focusing, energy - critical nonlinear Schrödinger problem in dimension $d = 4$ for initial data below a ground state threshold

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**Abstract:** In this paper we prove global well - posedness and scattering for the focusing, energy - critical nonlinear Schrödinger initial value problem in four dimensions. Previous work proved this in five dimensions and higher using the double Duhamel trick. In this paper, using long time Strichartz estimates we are able to overcome the logarithmic blowup in four dimensions.

## 1 Introduction

In this paper we study the Schrödinger initial value problem

$$\begin{aligned} iu_t + \Delta u &= F(u) = -|u|^2 u, \\ u(0, x) &= u_0 \in \dot{H}^1(\mathbf{R}^4). \end{aligned} \quad (1.1)$$

(1.1) belongs to a class of problems known as the focusing, nonlinear Schrödinger initial value problems,

$$\begin{aligned} iu_t + \Delta u &= F(u) = -|u|^p u, \\ u(0, x) &= u_0 \in \dot{H}^1(\mathbf{R}^d). \end{aligned} \quad (1.2)$$

(1.2) is called energy - critical if  $p = \frac{4}{d-2}$ ,  $d \geq 3$ . In general a solution to (1.2) conserves the quantities mass,

$$M(u(t)) = \int |u(t, x)|^2 dx = M(u(0)), \quad (1.3)$$

and energy,

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx - \frac{1}{p+2} \int |u(t, x)|^{p+2} dx = E(u(0)). \quad (1.4)$$

(1.2) is called energy - critical when  $p = \frac{4}{d-2}$  since a solution to (1.2) is invariant under the scaling

$$u(t, x) \mapsto \lambda^{\frac{d}{2}-1} u(\lambda^2 t, \lambda x), \quad (1.5)$$

and (1.5) preserves the energy (1.4).

There also exist the defocusing, energy - critical problems ( $F(u) = |u|^{\frac{4}{d-2}} u$ ), which are similar to the focusing problem in some ways, but also contain many important differences. The defocusing problem is now completely worked out.

**Theorem 1.1** *The defocusing initial value problem (1.2),  $F(u) = |u|^{\frac{4}{d-2}} u$ , is globally well - posed and scattering for all  $u_0 \in \dot{H}^1(\mathbf{R}^d)$ ,  $d \geq 3$ .*

**Definition 1.1 (Scattering)** *A solution  $u$  to (1.2),  $p = \frac{4}{d-2}$  is said to scatter forward in time if there exists  $u_+ \in \dot{H}^1$  such that*

$$\lim_{t \nearrow +\infty} \|u(t) - e^{it\Delta} u_+\|_{\dot{H}^1(\mathbf{R}^d)} = 0. \quad (1.6)$$

*Likewise,  $u$  is said to scatter backward in time if there exists  $u_- \in \dot{H}^1$  such that*

$$\lim_{t \searrow -\infty} \|u(t) - e^{it\Delta} u_-\|_{\dot{H}^1(\mathbf{R}^d)} = 0. \quad (1.7)$$

*Proof:* The proof of theorem 1.1 has involved contributions from a variety of authors. [11] proved theorem 1.1 for small data in both the focusing and defocusing problem. [11] also proved that (1.2) has a local solution for any initial data  $u_0 \in \dot{H}^1(\mathbf{R}^d)$ , where the time of existence depends on the size and profile of  $u_0$ .

For large data, the seminal result was the work of [4] and [5], proving theorem 1.1 for radial data in dimensions  $d = 3, 4$ , and also that for more regular  $u_0$ , this additional smoothness is preserved. See [21] for another proof of this last fact. [40] then extended theorem 1.1 to radial data in higher dimensions.

Then [13] extended theorem 1.1 to general  $u_0 \in \dot{H}^1$  when  $d = 3$ . Subsequently, [34] extended this to dimension  $d = 4$ , and [46], [47] extended theorem 1.1 to dimensions  $d \geq 5$ .  $\square$

**Remark:** [48] and [30] reproved theorem 1.1 in dimensions three and four using the long time Strichartz estimates of [14]. We will use long time Strichartz estimates similar to the estimates of [30] in this paper as well.

Returning to the focusing problem, we remark that theorem 1.1 does not hold for arbitrary data. In fact, by the virial identity (see for example [20])

$$\frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx = 8 \left[ \int |\nabla u(t, x)|^2 dx - \int |u(t, x)|^{\frac{2d}{d-2}} dx \right], \quad (1.8)$$

so for  $xu_0 \in L^2(\mathbf{R}^d)$  and  $E(u_0) < 0$ , the solution must break down in finite time. Moreover,

$$W(x) = W(x, t) = \frac{1}{\left(1 + \frac{|x|^2}{d(d-2)}\right)^{\frac{d-2}{2}}} \quad (1.9)$$

lies in  $\dot{H}^1(\mathbf{R}^d)$  and solves the elliptic equation

$$\Delta W + |W|^{\frac{4}{d-2}} W = 0. \quad (1.10)$$

Therefore, scattering cannot always occur even for global solutions. Instead, as in the mass - critical problem, we conjecture that scattering holds for initial data below the threshold given by (1.9).

**Conjecture 1.1** *Let  $d \geq 3$  and let  $u : I \times \mathbf{R}^d \rightarrow \mathbf{C}$  be a solution to (1.2),  $p = \frac{4}{d-2}$ . If*

$$\|u_0\|_{\dot{H}^1(\mathbf{R}^d)} < \|W\|_{\dot{H}^1(\mathbf{R}^d)}, \quad (1.11)$$

and

$$E(u_0) < E(W), \quad (1.12)$$

then

$$\int_I \int |u(t, x)|^{\frac{2(d+2)}{d-2}} dx dt \leq C(\|u_0\|_{\dot{H}^1}, E(u_0)) < \infty. \quad (1.13)$$

[10] and [11] proved that (1.2),  $p = \frac{4}{d-2}$  is well - posed on  $I$  for initial data  $u_0$  if and only if, for any  $J \subset I$  compact,  $S_J(u) < \infty$ . If  $S_{[t_1, \infty)}(u) < \infty$  for some  $t_1 \in \mathbf{R}$ , then  $u$  scatters forward in time. Likewise, if  $S_{(-\infty, t_1]}(u) < \infty$  then  $u$  scatters backward in time.

**Definition 1.2 (Scattering size)** *The scattering size of a solution to (1.2) on a time interval  $I$  is given by*

$$S_I(u) = \int_I \int_{\mathbf{R}^d} |u(t, x)|^{\frac{2(d+2)}{d-2}} dx dt. \quad (1.14)$$

**Definition 1.3 (Blow up)** *A solution  $u$  to (1.2) blows up forward in time on  $I$  if there exists  $t_1 \in I$  such that*

$$S_{[t_1, \sup(I))}(u) = \infty. \quad (1.15)$$

$u$  blows up backward in time if there exists  $t_1 \in I$  such that

$$S_{(\inf(I), t_1]}(u) = \infty. \quad (1.16)$$

Substantial progress has been made toward the proof of conjecture 1.1.

**Theorem 1.2** *Assume that  $E(u_0) < E(W)$ ,  $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ ,  $d = 3, 4, 5$ , and  $u_0$  is radial. Then (1.2) is globally well - posed and scatters forward and backward in time.*

*Proof:* See [23].  $\square$

Then [27] treated the nonradial case.

**Theorem 1.3** *Assume that  $E(u_0) < E(W)$ ,  $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ ,  $d \geq 5$ . Then (1.2) is globally well - posed and scatters forward and backward in time.*

*Proof:* See [27].  $\square$

**Remark:** The result of [27] was proved under the assumption that

$$\|u\|_{L_t^\infty \dot{H}_x^1(I \times \mathbf{R}^d)} < \|\nabla W\|_{L^2(\mathbf{R}^d)}. \quad (1.17)$$

Now by the energy trapping lemma of [23], if  $E(u_0) < E(W)$  and  $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ , (1.28) holds.

**Lemma 1.4** *If  $E(u_0) \leq (1 - \delta)E(W)$  and  $\|\nabla u_0\|_{L^2(\mathbf{R}^d)} < (1 - \delta)\|\nabla W\|_{L^2(\mathbf{R}^d)}$  for some  $\delta > 0$ , then there exists  $\bar{\delta}(\delta, d) > 0$  such that for all  $t \in I$ , where  $I$  is the maximal interval of existence of  $u$ ,*

$$\|\nabla u(t)\|_{L_x^2(\mathbf{R}^d)} \leq (1 - \bar{\delta})\|\nabla W\|_{L^2(\mathbf{R}^d)}. \quad (1.18)$$

*Proof:* This follows from the work of [1] and [38], which proved that if  $C_d$  is the best constant in the Sobolev embedding,

$$\|u\|_{L_x^{\frac{2d}{d-2}}(\mathbf{R}^d)} \leq C_d \|\nabla u\|_{L_x^2(\mathbf{R}^d)}, \quad (1.19)$$

and

$$\|u\|_{L_x^{\frac{2d}{d-2}}(\mathbf{R}^d)} = C_d \|\nabla u\|_{L_x^2(\mathbf{R}^d)}, \quad (1.20)$$

then  $u = CW_{\theta_0, x_0, \lambda_0}$  for some constant  $C$ ,  $\theta_0 \in \mathbf{R}$ ,  $x_0 \in \mathbf{R}^d$ , and  $\lambda_0 \in (0, \infty)$ , and

$$W_{\theta_0, x_0, \lambda_0} = \frac{1}{\lambda_0^{\frac{d-2}{2}}} e^{i\theta_0} W\left(\frac{x - x_0}{\lambda_0}\right), \quad (1.21)$$

$W$  is given by (1.9). In particular, when  $d = 4$ , (1.10) implies

$$0 = \langle \Delta W, W \rangle + \langle W, |W|^2 W \rangle = - \int |\nabla W|^2 dx + \int |W|^4 dx. \quad (1.22)$$

Then by (1.20),

$$C_4 = \frac{1}{\|W\|_{L_x^4(\mathbf{R}^4)}}, \quad (1.23)$$

so

$$E(W)(1 - \delta) \geq E(u_0) = \frac{1}{2} \int |\nabla u(t)|^2 dx \left(1 - \frac{1}{2} \frac{\|u(t)\|_{L_x^4(\mathbf{R}^4)}^2}{\|W\|_{L_x^4(\mathbf{R}^4)}^2}\right). \quad (1.24)$$

Now make a bootstrap argument. Since  $\|u(0)\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ , by local well-posedness

$$\|u(t)\|_{\dot{H}^1} \leq \|\nabla W\|_{L^2} \quad (1.25)$$

on some closed interval  $J$  of 0. Then by (1.20), (1.23), and (1.24),

$$E(W)(1 - \delta) = (1 - \delta) \frac{1}{4} \|W\|_{\dot{H}^1(\mathbf{R}^4)}^2 \geq \frac{1}{4} \|\nabla u(t)\|_{L_x^2(\mathbf{R}^4)}^2, \quad (1.26)$$

which in turn implies that  $\|u(t)\|_{\dot{H}^1(\mathbf{R}^4)}^2 \leq (1 - \delta) \|W\|_{\dot{H}^1(\mathbf{R}^4)}^2$ .  $\square$

Scattering results for the mass - critical problem ([32], [31], [42], [17]) assume that the initial data  $u_0$  has mass below the mass of a ground state. For the energy - critical problem it stands to reason that there should be two assumptions on the initial data because unlike the mass (1.3), the  $\dot{H}^1$  norm is not conserved. On the other hand, while energy is conserved, energy is not positive definite (1.4), so  $E(u(t)) < E(W)$  does not by itself give a bound on the size of  $u(t)$ . The author of this paper is personally unaware of any solutions  $u(t)$  to (1.2),  $p = \frac{4}{d-2}$  that satisfy (1.28) but not the initial conditions of theorem 1.2, although he suspects that there most likely are.

In this paper we prove global well - posedness and scattering for nonradial data in dimension four.

**Theorem 1.5** *Assume that  $E(u_0) < E(W)$ ,  $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ , and  $d = 4$ . Then (1.2) is globally well - posed and scatters forward and backward in time.*

As in [23] and [27], the proof uses the concentration compactness method.

**Theorem 1.6** *If (1.1) is not globally well - posed and scattering for all data satisfying  $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$  and  $E(u_0) < E(W)$ , then there exists a nonzero solution  $u$  to (1.1) on  $I$ , where  $I$  is the maximal interval of its existence, such that  $u$  is almost periodic for all  $t \in I$ .*

**Definition 1.4 (Almost periodicity)**  $u(t)$  is said to be almost periodic for all  $t \in I$  if there exists  $N(t) : I \rightarrow (0, \infty)$  and  $x(t) : I \rightarrow \mathbf{R}^4$  such that  $\frac{1}{N(t)}u(\frac{x-x(t)}{N(t)})$  lies in a compact set  $K \subset \dot{H}^1(\mathbf{R}^4)$  for all  $t \in I$ .

**Theorem 1.7** *The only almost periodic solution to (1.1) on the maximal interval of its existence  $I$ , with  $\|\nabla u(t)\|_{L_t^\infty L_x^2(I \times \mathbf{R}^4)} < \|\nabla W\|_{L^2}$ , is  $u \equiv 0$ .*

Then to prove theorem 1.5, it suffices to show theorems 1.6 and 1.7. In fact,

**Theorem 1.8** *To prove theorem 1.7 it suffices to show that the only global, almost periodic solution to (1.1) on  $\mathbf{R}$  with*

$$N(t) \geq 1, \quad N(0) = 1, \quad (1.27)$$

*is  $u \equiv 0$ .*

The main difference between [27] and this result in dimension  $d = 4$  is that in dimensions  $d \geq 5$  the dispersive estimate (2.15) is doubly integrable, allowing [27] to make use of the double Duhamel trick. However, here, even though we can prove  $u \in L_t^\infty L_x^3$ , and thus  $F(u) \in L^1$ , the double integral of (2.18) diverges logarithmically. Nevertheless, this logarithmically divergent result is good enough to be used in an interaction Morawetz estimate, proving theorem 1.8.

**Outline of Proof:** In §2, some linear estimates and harmonic analysis results will be discussed. These results will be used frequently throughout the rest of the paper. Only one of the results in this section is new.

In §3, the concentration compactness method will be discussed, sketching [23] and then [27]’s proof of theorems 1.6. We will also discuss almost periodic solutions to (1.1) and sketch [27]’s proof of 1.8. Finally we will bound the  $L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)$  norm of a solution satisfying (1.27). In §4 we prove the long time Strichartz estimate. In contrast to [48] and [29], we will consider the quantity

$$\int_I \frac{1}{N(t)^2} dt. \quad (1.28)$$

The long time Strichartz estimates allow us to easily exclude the case when  $\int_{\mathbf{R}} N(t)^{-2} dt < \infty$ . In §5 we show that the soliton blowup solution, that is  $N(t) \equiv 1$ , is  $u \equiv 0$ . Finally, in §6 we will extend this argument to a quasi soliton solution,  $(1.28) = \infty$ . This completes the proof of theorem 1.5.

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## 2 Linear Estimates and harmonic analysis

In this section we describe the tools from harmonic analysis that will be used in this paper. None of the results of this section, with the exception of theorem 2.6, are new. Theorem 2.6 was proved by [30] for dimension  $d = 3$  only.

**Definition 2.1 (Fourier transform)** *Suppose  $f \in L^1(\mathbf{R}^d)$ . Then*

$$\mathcal{F}f(\xi) = (2\pi)^{-d/2} \int e^{-ix \cdot \xi} f(x) dx. \quad (2.1)$$

*The inverse Fourier transform is then given by*

$$\mathcal{F}^{-1}\hat{f}(x) = (2\pi)^{-d/2} \int e^{ix \cdot \xi} \hat{f}(\xi) d\xi. \quad (2.2)$$

Plancherel's theorem proved that the Fourier transform and inverse Fourier transform provide a unitary transformation between functions in  $L_x^2(\mathbf{R}^d)$  and functions in  $L_\xi^2(\mathbf{R}^d)$ . Because of this fact it is useful to decompose a function via a partition of unity in Fourier space, or a Littlewood - Paley decomposition.

**Definition 2.2 (Littlewood - Paley decomposition)** *Let  $\phi \in C_0^\infty(\mathbf{R}^d)$  be a radial, decreasing function,  $\phi(x) = 1$  for  $|x| \leq 1$ ,  $\phi(x)$  is supported on  $|x| > 2$ . Then for any  $j \in \mathbf{Z}$  let*

$$P_j f = (2\pi)^{-d/2} \int e^{ix \cdot \xi} [\phi(2^{-j-1}\xi) - \phi(2^{-j}\xi)] \hat{f}(\xi) d\xi. \quad (2.3)$$

**Remark:** It is often convenient to write  $P_N$ , which is given by the multiplier

$$[\phi(\frac{1}{N}\xi) - \phi(\frac{1}{2N}\xi)], \quad (2.4)$$

or to sum over  $N \geq M$ , which in this case would be over  $M = 2^j N$ ,  $j \geq 0$ .

To simplify notation we often write  $u_k$  or  $u_N$  instead of  $P_k u$  or  $P_N u$ .

**Theorem 2.1 (Littlewood - Paley theorem)** *For any  $1 < p < \infty$ ,*

$$\|(\sum_j |P_j f|^2)^{1/2}\|_{L_x^p(\mathbf{R}^d)} \sim_{p,d} \|f\|_{L^p(\mathbf{R}^d)}. \quad (2.5)$$

*Proof:* This is a well - known fact from harmonic analysis. See [35], [36], [43], or many other sources.  $\square$

The proof of theorem 2.1 utilizes the maximal function, which can be defined in any dimension. We will use the maximal function in one dimension only.

**Definition 2.3 (Maximal function)** For a function  $f \in L^p(\mathbf{R})$ ,  $1 \leq p \leq \infty$ ,

$$\mathcal{M}(f)(x) = \sup_{T>0} \frac{1}{T} \int_{x-T}^{x+T} |f(t)| dt. \quad (2.6)$$

**Theorem 2.2 (Maximal theorem)** For any  $1 < p \leq \infty$ ,

$$\|\mathcal{M}(f)\|_{L^p(\mathbf{R})} \lesssim_p \|f\|_{L^p(\mathbf{R})}. \quad (2.7)$$

*Proof:* See [35], [36], or [43]. The proof there is described in any dimension.  $\square$

**Theorem 2.3 (Sobolev embedding)** For  $1 \leq p \leq q \leq \infty$ ,

$$\|P_j f\|_{L^q(\mathbf{R}^d)} \lesssim 2^{jd(\frac{1}{p}-\frac{1}{q})} \|P_j f\|_{L^p(\mathbf{R}^d)}. \quad (2.8)$$

*Proof:* See for example [45].  $\square$

**Lemma 2.4 (Bernstein's lemma)** For any  $s \in \mathbf{R}$ ,  $j \in \mathbf{Z}$ ,  $1 < p < \infty$ ,

$$\|P_j f\|_{L^p(\mathbf{R}^d)} \sim_{p,d} \| |\nabla|^s f \|_{L^p(\mathbf{R}^d)}. \quad (2.9)$$

*Proof:* See [44].  $\square$

Theorem 2.1, theorem 2.3, and lemma 2.4 will be used throughout this paper, frequently in conjunction with one another.

The Fourier transform is extremely useful to the study of the linear Schrödinger problem,

$$(i\partial_t + \Delta)u = F, \quad u(0, x) = u_0, \quad (2.10)$$

because the solution to (2.10) when  $F = 0$  is given by

$$e^{it\Delta}u_0 = (2\pi)^{-d/2} \int e^{-it|\xi|^2} e^{ix \cdot \xi} \hat{f}(\xi) d\xi, \quad (2.11)$$

and the general strong solution to (2.10) is given by

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) - i \int_{t_0}^t e^{i(t-\tau)\Delta}F(\tau) d\tau. \quad (2.12)$$



Since  $|e^{it|\xi|^2}| = 1$ ,

$$\|e^{it\Delta}f\|_{L^2(\mathbf{R}^d)} = \|f\|_{L^2(\mathbf{R}^d)}, \quad (2.13)$$

and in fact, for any  $L^2$  - based Sobolev space,

$$\|e^{it\Delta}f\|_{\dot{H}_x^s(\mathbf{R}^d)} = \|f\|_{\dot{H}_x^s(\mathbf{R}^d)}. \quad (2.14)$$

By completing the square in the exponent of (2.11) and stationary phase computations,

$$e^{it\Delta}f(x) = \frac{1}{(4\pi t)^{d/2}} e^{-id\pi/4} \int e^{-i\frac{|x-y|^2}{4t}} f(y) dy. \quad (2.15)$$

**Remark:** More generally, if  $P$  is any Fourier multiplier, that is,

$$Pf(x) = \mathcal{F}^{-1}(P\mathcal{F}f)(\xi), \quad (2.16)$$

then

$$Pf(x) = \int (\mathcal{F}^{-1}P)(x-y)f(y)dy. \quad (2.17)$$

Therefore,

$$\|e^{it\Delta}f\|_{L_x^\infty(\mathbf{R}^d)} \lesssim_d t^{-d/2} \|f\|_{L^1(\mathbf{R}^d)}. \quad (2.18)$$

Using both analysis on the Fourier side (2.11) (see [37]), and on the spatial side (2.15) (see [19], [22], and [49]), we have the sharp result

**Theorem 2.5 (Strichartz estimates)** *For  $d \geq 3$ , and  $(p_1, q_1), (p_2, q_2)$  satisfying  $p_j \geq 2$ ,*

$$\frac{2}{p_j} = d\left(\frac{1}{2} - \frac{1}{q_j}\right), \quad (2.19)$$

*if  $u$  solves (2.10) on  $I$ ,  $t_0 \in I$ , and  $\frac{1}{p'} = 1 - \frac{1}{p}$ , then*

$$\|u\|_{L_t^{p_1} L_x^{q_1}(I \times \mathbf{R}^d)} \lesssim_d \|u(t_0)\|_{L_x^2(\mathbf{R}^d)} + \|F\|_{L_t^{p'_2} L_x^{q'_2}(I \times \mathbf{R}^d)}. \quad (2.20)$$

*Proof:* See [37] for the seminal result, [19] and [49] for the non - endpoint results ( $p_j > 2$ ), and [22] for the endpoint case. See [41] for a nice overview of this work.  $\square$

We will also utilize the maximal Strichartz estimate of [30], which was introduced to provide a new proof of global well - posedness and scattering for the defocusing, three - dimensional energy - critical problem.

**Theorem 2.6 (Maximal Strichartz estimate)** *Suppose that  $t, t_0 \in I$ , and*

$$v(t) = \int_{t_0}^t e^{i(t-\tau)\Delta} F(\tau) d\tau. \quad (2.21)$$

*Then for any  $d \geq 3$ ,  $q > \frac{2d}{d-2}$ ,*

$$\| \sup_j 2^{j(\frac{d}{q} - (d-2))} \| P_j v(t) \|_{L_x^q(\mathbf{R}^d)} \|_{L_t^2(I)} \lesssim \| F \|_{L_t^2 L_x^1(I \times \mathbf{R}^d)}. \quad (2.22)$$

*Proof:* This is proved by combining the dispersive estimate (2.18) with the Sobolev embedding theorem (theorem 2.3). If  $q > \frac{2d}{d-2}$  then  $d(\frac{1}{2} - \frac{1}{q}) > 1$ , so

$$\begin{aligned} & 2^{j(\frac{d}{q} - (d-2))} \int_{|t-\tau| > 2^{-2j}} \frac{1}{(t-\tau)^{d(\frac{1}{2} - \frac{1}{q})}} \| P_j F(u(\tau)) \|_{L_x^{q'}(\mathbf{R}^d)} d\tau \\ & \lesssim \sum_{k \geq 0} 2^{-kd(\frac{1}{2} - \frac{1}{q})} 2^{2j} \int_{|t-\tau| \sim 2^k 2^{-2j}} \| F(\tau) \|_{L_x^1(\mathbf{R}^d)} d\tau \lesssim_q \mathcal{M}(\| F(\tau) \|_{L_x^1(\mathbf{R}^d)})(t). \end{aligned} \quad (2.23)$$

Also by Sobolev embedding

$$\begin{aligned} & 2^{j(\frac{d}{q} - (d-2))} \left\| \int_{|t-\tau| \leq 2^{-2j}} P_j e^{i(t-\tau)\Delta} F(u(\tau)) d\tau \right\|_{L_x^q(\mathbf{R}^d)} \\ & \lesssim 2^{2j} \int_{|t-\tau| \leq 2^{-2j}} \| F(\tau) \|_{L_x^1(\mathbf{R}^d)} d\tau \lesssim \mathcal{M}(\| F(\tau) \|_{L_x^1(\mathbf{R}^d)})(t). \end{aligned} \quad (2.24)$$

Therefore,

$$2^{j(\frac{d}{q} - (d-2))} \| P_j v(t) \|_{L_x^q(\mathbf{R}^d)} \lesssim \mathcal{M}(\| F(\tau) \|_{L_x^1(\mathbf{R}^d)})(t), \quad (2.25)$$

so by theorem 2.2 the proof is complete.  $\square$

We conclude the section by discussing the double Duhamel trick. This technique was introduced in [13] to study the defocusing, energy - critical Schrödinger initial value problem when  $d = 3$ , and in [27] for the focusing energy - critical problem for dimensions  $d \geq 5$ . See also [39]. The double Duhamel trick is also used to study wave ([6], [7], [8], [28], [29]) and KdV ([18]) problems.

Suppose  $I = [t_-, t_+]$  and  $u$  solves the equation

$$(i\partial_t + \Delta)u = F(t) + G(t). \quad (2.26)$$

Then by (2.12), for any  $t \in I$ ,

$$\begin{aligned}
& e^{i(t-t_-)\Delta}u(t_-) - i \int_{t_-}^t e^{i(t-s_-)\Delta}F(s_-)ds_- - i \int_{t_-}^t e^{i(t-s_-)\Delta}G(s_-)ds_- = u(t) \\
& = e^{i(t-t_+)\Delta}u(t_+) - i \int_{t_+}^t e^{i(t-s_+)\Delta}F(s_+)ds_+ - i \int_{t_+}^t e^{i(t-s_+)\Delta}G(s_+)ds_+.
\end{aligned} \tag{2.27}$$

Then if  $X$  is some Hilbert space, such as  $L^2(\mathbf{R}^d)$  or the weighted  $L^2$  space that we will use in this paper, then by simple linear algebra, for  $A + B = A' + B'$ ,

$$\langle A + B, A' + B' \rangle \lesssim |A|^2 + |A'|^2 + \langle B, B' \rangle, \tag{2.28}$$

so then

$$\begin{aligned}
\|u(t)\|_X^2 & \lesssim \|e^{i(t-t_-)\Delta}u(t_-)\|_X^2 + \|e^{i(t-t_+)\Delta}u(t_+)\|_X^2 + \left\| \int_{t_-}^t e^{i(t-s_-)\Delta}F(s_-)ds_- \right\|_X^2 \\
& + \left\| \int_{t_+}^t e^{i(t-s_+)\Delta}F(s_+)ds_+ \right\|_X^2 + \left\langle \int_{t_-}^t e^{i(t-s_-)\Delta}G(s_-)ds_-, e^{i(t-s_+)\Delta}G(s_+)ds_+ \right\rangle_X.
\end{aligned} \tag{2.29}$$

### 3 Concentration compactness

In this section we briefly discuss some concentration compactness results that will be used in this paper.

*Sketch of the proof of theorem 1.6:* The reader should consult [23] or [27] for a complete treatment of the concentration compactness method. Recall that the hypotheses of theorem 1.6 imply that there exists  $E_* < \|\nabla W\|_{L^2(\mathbf{R}^4)}$  such that

$$\|u(t)\|_{L_t^\infty \dot{H}_x^1(I \times \mathbf{R}^4)} \leq E_*, \tag{3.1}$$

where  $I$  is the maximal interval of existence for a solution to (1.1). Now let

$$C(E) = \sup\left\{ \|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbf{R} \times \mathbf{R}^d)} : \|u\|_{L_t^\infty \dot{H}_x^1(\mathbf{R} \times \mathbf{R}^d)} \leq E \right\}. \tag{3.2}$$

By the results of [11],  $C(E) \lesssim E$  for  $E$  small. Moreover, by a stability result in  $d \geq 5$  (see [27]) and a simple calculation in dimensions  $d = 3, 4$ ,  $C(E)$  is a continuous function of  $E$ . Therefore, if there exists a non-scattering solution to (1.1) satisfying (3.1), then by the continuity of  $C(E)$ , there exists  $E_* < \|\nabla W\|_{L^2}$  such that  $C(E_*) = \infty$  and  $C(E) < \infty$  for all  $E < E_*$ .

Now take a sequence  $u_n(t)$  of solutions to (1.1) such that

$$\|u_n(t)\|_{L_t^\infty \dot{H}_x^1(\mathbf{R} \times \mathbf{R}^d)} \nearrow E_*, \tag{3.3}$$

and

$$S_{[0,\infty)}(u_n) = S_{(-\infty,0]}(u_n) = n. \quad (3.4)$$

Then [24] proved that  $u_n(0)$  can be decomposed into asymptotically decoupling profiles, such that for any  $J$ ,

$$u_n(0) = \sum_{j=1}^J g_n^j e^{it_n^j \Delta} \phi^j + w_n^J, \quad (3.5)$$

where  $g_n^j$  is an element of a group generated by scaling and translation symmetries,  $w_n^J$  represents an error, and the group elements  $g_n^j$  asymptotically decouple. The asymptotic decoupling implies that if  $u^j(t)$  is the solution to (1.1) with initial data given by  $\phi^j$ , then (3.4) implies that for one  $j_0$ ,  $t_n^{j_0} \rightarrow 0$  and

$$\|u^{j_0}(t)\|_{L_t^\infty \dot{H}_x^1(I \times \mathbf{R}^d)} = E_*, \quad (3.6)$$

where  $I$  is the maximal interval of existence for  $u^{j_0}(t)$ , all other  $\phi^j = 0$ , and

$$\|u^{j_0}(t)\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I \times \mathbf{R}^d)} = \infty, \quad (3.7)$$

where  $I$  is the maximal interval of existence of  $u^{j_0}$ . Making the above argument again for a sequence  $u^{j_0}(t_n)$ ,  $t_n \in I$  shows that  $u^{j_0}(t_n)$  has a subsequence that converges in  $\dot{H}^1/G$ , where  $G$  is the group of symmetries  $g_n^j$ . This proves theorem 1.6.  $\square$

Notice that by the Arzela - Ascoli theorem, if  $u$  is an almost periodic solution to (1.1), then there exists  $x(t) : I \rightarrow \mathbf{R}^d$  and  $N(t) : I \rightarrow (0, \infty)$ , such that for any  $\eta > 0$  there exists  $C(\eta) < \infty$  such that

$$\int_{|x-x(t)| > \frac{C(\eta)}{N(t)}} |\nabla u(t, x)|^2 dx + \int_{|\xi| > C(\eta)N(t)} |\xi|^2 |\hat{u}(t, \xi)|^2 d\xi + \int_{|\xi| < \frac{1}{C(\eta)N(t)}} |\xi|^2 |\hat{u}(t, \xi)|^2 d\xi < \eta. \quad (3.8)$$

Moreover, (see [26] for a proof)

$$|N'(t)| \lesssim N(t)^3, \quad (3.9)$$

and

$$\int_I N(t)^2 dt \lesssim \int_I \int |u(t, x)|^{\frac{2(d+2)}{d-2}} dx dt \lesssim \int_I N(t)^2 dt + 1. \quad (3.10)$$

Making use of (3.9), [27] proved theorem 1.8.

*Sketch of proof of theorem 1.8:* Suppose  $u(t)$  is an almost periodic solution to (1.1). Then [27] showed that one can take a limit of  $u(t_n)$  in  $\dot{H}^1/G$  and obtain a solution to (1.1) satisfying either

$$N(t) \geq 1, \quad t \in \mathbf{R}, \quad N(0) = 1, \quad (3.11)$$

or than  $u$  blows up in finite time. However, finite time blowup fails to occur due to concentration compactness and conservation of mass (1.3). Indeed, by (3.10), if  $u$  blows up in finite time, say at  $T = 0$ ,  $N(t) \nearrow \infty$  as  $t \searrow 0$ . Let  $\psi \in C_0^\infty(\mathbf{R}^d)$  be a radial function,  $\psi = 1$  on  $|x| \leq 1$ ,  $\psi$  supported on  $|x| \leq 2$ . By (3.8) and Hölder's inequality, for any  $R > 0$ ,

$$\lim_{t \searrow 0} \int \psi\left(\frac{x}{R}\right)^2 |u(t, x)|^2 dx = \lim_{t \searrow 0} M_R(t) = 0. \quad (3.12)$$

Moreover, integrating by parts,

$$\frac{d}{dt} M_R(t) \leq \frac{1}{R} \psi'\left(\frac{x}{R}\right) \psi\left(\frac{x}{R}\right) |\nabla u(t, x)| |u(t, x)| dx \leq \frac{1}{R} M_R(t)^{1/2} \|\nabla u(t)\|_{L_x^2(\mathbf{R}^d)}. \quad (3.13)$$

Therefore, (3.12) combined with the fundamental theorem of calculus and (3.13) implies that  $\int |u(t, x)|^2 dx = 0$  for any  $t > 0$ . However, this implies  $u \equiv 0$ , which contradicts  $u$  blowing up in finite time.  $\square$

**Theorem 3.1** *If  $u(t)$  is an almost periodic solution to (1.1) satisfying  $N(t) \geq 1$ , then*

$$\|u(t)\|_{L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)} < \infty. \quad (3.14)$$

**Remark:** This is an endpoint of a more general result of [33].

**Remark:** From now on since we are considering an almost periodic solution  $u$  to (1.1), let  $A \lesssim B$  denote  $A \leq C(u)B$ .

*Proof:* By the Duhamel formula (2.12), for any  $t_0 \in \mathbf{R}$ ,

$$u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau. \quad (3.15)$$

Now by (3.8), for a fixed  $t$ ,

$$e^{i(t-t_0)\Delta} u(t_0) \rightharpoonup 0 \quad (3.16)$$

weakly in  $\dot{H}^1$  as  $t_0 \rightarrow \pm\infty$ . Indeed, this follows from (3.8) if  $N(t_0) \nearrow +\infty$  or  $\searrow 0$ . If  $N(t_0) \sim 1$  then this follows by combining (3.8) and the dispersive estimate (2.18). Therefore, for any  $j \in \mathbf{Z}$ ,

$$\|P_j u(t)\|_{L_x^\infty(\mathbf{R}^4)} \lesssim \lim_{t_0 \rightarrow \infty} \left\| \int_{t_0}^t e^{i(t-\tau)\Delta} P_j F(u(\tau)) d\tau \right\|_{L_x^\infty(\mathbf{R}^4)}. \quad (3.17)$$

Now by Sobolev embedding (2.8) and (2.13),

$$\left\| \int_{t-2^{-2j}}^t e^{i(t-\tau)\Delta} P_j F(u(\tau)) d\tau \right\|_{L_x^\infty(\mathbf{R}^4)} \lesssim 2^{2j} \|u\|_{L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)}^3. \quad (3.18)$$

Also by the dispersive estimate (2.18),

$$\begin{aligned} & \left\| \int_{t_0}^{t-2^{-2j}} e^{i(t-\tau)\Delta} P_j F(u(\tau)) d\tau \right\|_{L_x^\infty(\mathbf{R}^4)} \\ & \lesssim \|u\|_{L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)}^3 \int_{t > 2^{-2j}} \frac{1}{t^2} dt \lesssim 2^{2j} \|u\|_{L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)}^3. \end{aligned} \quad (3.19)$$

Then by (3.8), if  $j_0(\eta)$  is the largest integer such that  $2^{j_0} \leq \frac{1}{C(\eta)}$ , then by  $N(t) \geq 1$ , (3.18) and (3.19)

$$\begin{aligned} \|P_{\leq j_0(\eta)} u(t)\|_{L_x^3(\mathbf{R}^4)}^3 & \lesssim \sum_{k_1 \leq k_2 \leq k_3 \leq j_0(\eta)} \|P_{k_1} u(t)\|_{L_x^\infty(\mathbf{R}^4)} \|P_{k_2} u(t)\|_{L_x^2(\mathbf{R}^4)} \|P_{k_3} u(t)\|_{L_x^2(\mathbf{R}^4)} \\ & \lesssim \|u\|_{L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)}^3 \sum_{k_1 \leq k_2 \leq k_3 \leq j_0(\eta)} 2^{2k_1} 2^{-k_2} 2^{-k_3} \|\nabla P_{k_2} u(t)\|_{L_x^2(\mathbf{R}^4)} \|\nabla P_{k_3} u(t)\|_{L_x^2(\mathbf{R}^4)} \\ & \lesssim \eta^2 \|u\|_{L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)}^3. \end{aligned} \quad (3.20)$$

Meanwhile, by Bernstein's inequality, since  $\dot{H}^{2/3}(\mathbf{R}^4) \subset L_x^3(\mathbf{R}^4)$ ,

$$\|P_{> j_0} u(t)\|_{L_x^3(\mathbf{R}^4)}^3 \lesssim 2^{-j_0}, \quad (3.21)$$

so

$$\|u(t)\|_{L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)} \lesssim 2^{-j_0(\eta)/3} + \eta^{2/3} \|u(t)\|_{L_t^\infty L_x^3(\mathbf{R} \times \mathbf{R}^4)}. \quad (3.22)$$

□

## 4 Long Time Strichartz estimate

Now we prove a long time Strichartz estimate. Long time Strichartz estimates were introduced in [14] to study the mass - critical nonlinear Schrödinger equation in dimensions  $d \geq 3$ . Subsequently, [48] and [30] utilized long - time Strichartz estimates for the defocusing, energy - critical nonlinear Schrödinger problem in dimensions  $d = 4$  and  $d = 3$  respectively. The long - time Strichartz estimates of  $d = 3$  relied on the maximal Strichartz estimates. Here we prove the long - time Strichartz estimates of [30], modified to the case when  $d = 4$ . In this case the crucial quantity is

$$K = \int_I N(t)^{-2} dt. \quad (4.1)$$

The quantity  $\int_I \frac{1}{N(t)} dt$  was quite useful in the defocusing case since it scaled like the interaction Morawetz estimates of [12], [13], and [42]. However, in the focusing case there is no such estimate. On the other hand, (1.28) is a lower bound for

$$\int_I \int |u(t, x)|^2 dx dt, \quad (4.2)$$

which allows us to prove some very useful interaction Morawetz estimates.

**Theorem 4.1 (Long time Strichartz estimate)** *For any  $j$ ,*

$$\left( \sum_{k \leq j} \|\nabla u_j\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)}^2 \right)^{1/2} + 2^{2j} \left\| \sup_{k \geq j} 2^{-2k} \|u_k(t)\|_{L_x^\infty(\mathbf{R}^4)} \right\|_{L_t^2(I)} \lesssim (1 + K 2^{4j})^{1/2}. \quad (4.3)$$

*Proof:* By Sobolev embedding and Bernstein's inequality

$$2^{4j} \left( \sum_{k \geq j} 2^{-4k} \|e^{i(t-t_0)\Delta} P_k u(t_0)\|_{L_t^2 L_x^\infty(I \times \mathbf{R}^4)}^2 \right) \lesssim 2^{2j} \|P_{>j} u(t_0)\|_{L_x^2(\mathbf{R}^4)}^2 \lesssim \|u(t_0)\|_{\dot{H}^1(\mathbf{R}^4)}^2 \lesssim 1, \quad (4.4)$$

and

$$\sum_{k \leq j} \|\nabla e^{i(t-t_0)\Delta} P_k u(t_0)\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)}^2 \lesssim \|\nabla u(t_0)\|_{L_x^2(\mathbf{R}^4)}^2 \lesssim 1. \quad (4.5)$$

Now let

$$\begin{aligned} \|u\|_{Y(I \times \mathbf{R}^4)} &= \sup_j 2^{2j} (1 + K 2^{4j})^{-1/2} \left\| \sup_{k \geq j} 2^{-2k} \|u_j(t)\|_{L_x^\infty(\mathbf{R}^4)} \right\|_{L_t^2(I)} \\ &\quad + \sup_j (1 + K 2^{4j})^{-1/2} \left( \sum_{k \leq j} 2^{2k} \|u_k(t)\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)}^2 \right)^{1/2}. \end{aligned} \quad (4.6)$$

By conservation of energy and Bernstein's inequality,

$$\begin{aligned} &\|P_{\leq cN(t)} P_{\geq j} u\|_{L_t^6 L_x^3(I \times \mathbf{R}^4)}^3 \\ &\lesssim \left\| \sum_{j \leq k_1 \leq k_2 \leq k_3} \|P_{\leq cN(t)} u_{k_1}\|_{L_x^\infty(\mathbf{R}^4)} \|P_{\leq cN(t)} u_{k_2}\|_{L_x^2(\mathbf{R}^4)} \|P_{\leq cN(t)} u_{k_3}\|_{L_x^2(\mathbf{R}^4)} \right\|_{L_t^2(I)} \\ &\lesssim \eta^2 \left\| \sup_{k \geq j} 2^{-2k} \|u_k(t)\|_{L_x^\infty(\mathbf{R}^4)} \right\|_{L_t^2(I)}. \end{aligned} \quad (4.7)$$

Also, by Bernstein's inequality

$$\|P_{\geq cN(t)}u\|_{L_t^6 L_x^3(I \times \mathbf{R}^4)}^3 \lesssim \left( \int_I c^{-2} N(t)^{-2} dt \right)^{1/2} \lesssim c^{-1} K^{1/2}. \quad (4.8)$$

Therefore,

$$\|u_{\geq j}^3\|_{L_t^2 L_x^1(I \times \mathbf{R}^4)} \lesssim c^{-1} K^{1/2} + \eta^2 (1 + 2^{4j} K)^{1/2} 2^{-2j} \|u\|_{Y(I \times \mathbf{R}^4)}, \quad (4.9)$$

and by theorem 2.6 and Sobolev embedding,

$$\begin{aligned} & \left\| \sup_{k \geq j} \|P_k \int_{t_0}^t e^{i(t-\tau)\Delta} F(u_{\geq j}) d\tau\|_{L_x^\infty(\mathbf{R}^4)} \|L_t^2(I)\right\| \\ & \lesssim c^{-1} K^{1/2} + \eta^2 (1 + 2^{4j} K)^{1/2} 2^{-2j} \|u\|_{Y(I \times \mathbf{R}^4)}, \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} & \left( \sum_{k \leq j} 2^{2k} \|P_k \int_{t_0}^t e^{i(t-\tau)\Delta} F(u_{\geq j}) d\tau\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)}^2 \right)^{1/2} \\ & \lesssim c^{-1} 2^{2j} K^{1/2} + \eta^2 (1 + 2^{4j} K)^{1/2} \|u\|_{Y(I \times \mathbf{R}^4)}. \end{aligned} \quad (4.11)$$

Next, by the Littlewood - Paley theorem and Sobolev embedding,

$$\begin{aligned} & \|u_{\leq j}\|_{L_{t,x}^6(I \times \mathbf{R}^4)} \lesssim \|\nabla u_{\leq j}\|_{L_t^6 L_x^{12/5}(I \times \mathbf{R}^4)} \\ & \lesssim \left( \sum_{k \leq j} 2^{2k} \|u_k\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)}^2 \right)^{1/6} \|\nabla u\|_{L_t^\infty L_x^2(I \times \mathbf{R}^4)}^{2/3} \lesssim (1 + 2^{4j} K)^{1/6} \|u\|_{Y(I \times \mathbf{R}^4)}^{1/3}. \end{aligned} \quad (4.12)$$

Therefore by Sobolev embedding, (4.9) and (4.12),

$$\begin{aligned} & \|(u_{\geq j}^2 u_{\leq j})\|_{L_t^2 L_x^{4/3}(I \times \mathbf{R}^4)} \lesssim \|u_{\geq j}\|_{L_t^6 L_x^3(I \times \mathbf{R}^4)}^2 \|u_{\leq j}\|_{L_t^6 L_x^{12}(I \times \mathbf{R}^4)} \\ & \lesssim 2^{j/3} (c^{-1} K^{1/2} + \eta^2 2^{-2j} (1 + 2^{4j} K)^{1/2} \|u\|_{Y(I \times \mathbf{R}^4)})^{2/3} (1 + 2^{4j} K)^{1/6} \|u\|_{Y(I \times \mathbf{R}^4)}^{1/3} \\ & \lesssim 2^{j/3} c^{-2/3} K^{1/3} (1 + 2^{4j} K)^{1/6} \|u\|_{Y(I \times \mathbf{R}^4)}^{1/3} + 2^{-j} \eta^{4/3} (1 + 2^{4j} K)^{1/2} \|u\|_{Y(I \times \mathbf{R}^4)}. \end{aligned} \quad (4.13)$$

(4.13) implies, by Sobolev embedding, that

$$\begin{aligned} & \left\| \sup_{k \geq j} 2^{-2k} \|P_k \int_{t_0}^t e^{i(t-\tau)\Delta} O(u_{\geq j}^2 u_{\leq j}) d\tau\|_{L_x^\infty(\mathbf{R}^4)} \|L_t^2(I)\right\| \\ & \lesssim 2^{-j} \left\| \int_{t_0}^t e^{i(t-\tau)\Delta} O(u_{\geq j}^2 u_{\leq j}) d\tau\right\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)} \\ & \lesssim 2^{-2j/3} c^{-2/3} K^{1/3} (1 + 2^{4j} K)^{1/6} \|u\|_{Y(I \times \mathbf{R}^4)}^{1/3} \\ & \quad + 2^{-2j} \eta^{4/3} (1 + 2^{4j} K)^{1/2} \|u\|_{Y(I \times \mathbf{R}^4)}, \end{aligned} \quad (4.14)$$

and by Strichartz estimates



$$\begin{aligned}
& \left( \sum_{k \leq j} 2^{2k} \|P_k \int_{t_0}^t e^{i(t-\tau)\Delta} O(u_{\leq j}^2 u_{\leq j}) d\tau\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)}^2 \right)^{1/2} \lesssim 2^j \|u_{\leq j}^2 u_{\leq j}\|_{L_t^2 L_x^{4/3}(I \times \mathbf{R}^4)} \\
& \lesssim c^{-2/3} 2^{4j/3} K^{1/3} (1 + 2^{4j} K)^{1/6} \|u\|_{Y(I \times \mathbf{R}^4)}^{1/3} + \eta^{4/3} (1 + 2^{4j} K)^{1/2} \|u\|_{Y(I \times \mathbf{R}^4)}.
\end{aligned} \tag{4.15}$$

Next, by Sobolev embedding,

$$\begin{aligned}
\|(P_{\leq cN(t)} u_{\leq j})^2\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)} & \lesssim \|\nabla u_{\leq j}\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)} \|u_{\leq cN(t)}\|_{L_t^\infty L_x^4(I \times \mathbf{R}^4)} \\
& \lesssim \eta (1 + 2^{4j} K)^{1/2} \|u\|_{Y(I \times \mathbf{R}^4)},
\end{aligned} \tag{4.16}$$

and by Bernstein's inequality and Sobolev embedding

$$\|(P_{\leq cN(t)} u_{\leq j})^2\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)} \lesssim 2^{2j} \left( \int \|u_{> cN(t)}\|_{L_x^2(\mathbf{R}^4)}^2 dt \right)^{1/2} \lesssim c^{-1} K^{1/2} 2^{2j}. \tag{4.17}$$

(4.16) and (4.17) imply that

$$\begin{aligned}
& \|\nabla(u_{\leq j}^2 u_{> j})\|_{L_t^2 L_x^{4/3}(I \times \mathbf{R}^4)} + \|\nabla u_{\leq j}^3\|_{L_t^2 L_x^{4/3}(I \times \mathbf{R}^4)} \\
& \lesssim \|\nabla u\|_{L_t^\infty L_x^2(I \times \mathbf{R}^4)} \|u_{\leq j}^2\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)} \lesssim c^{-1} K^{1/2} 2^{2j} + \eta (1 + 2^{4j} K)^{1/2} \|u\|_{Y(I \times \mathbf{R}^4)}.
\end{aligned} \tag{4.18}$$

Therefore,

$$\begin{aligned}
& \left( \sum_{k \leq j} 2^{2k} \|P_k \int_{t_0}^t e^{i(t-\tau)\Delta} O(u_{\leq j}^2 u) d\tau\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)}^2 \right)^{1/2} \\
& \lesssim c^{-1} K^{1/2} 2^{2j} + \eta (1 + 2^{4j} K)^{1/2} \|u\|_{Y(I \times \mathbf{R}^4)}.
\end{aligned} \tag{4.19}$$

Also, by Sobolev embedding, Bernstein's inequality, and (4.19),

$$\begin{aligned}
& \left\| \sup_{k \geq j} 2^{-2k} \|P_k \int_{t_0}^t e^{i(t-\tau)\Delta} O(u_{\leq j}^2 u) d\tau\|_{L_x^\infty(\mathbf{R}^4)} \right\|_{L_t^2(I)} \\
& \lesssim c^{-1} K^{1/2} + \eta 2^{-2j} (1 + 2^{4j} K)^{1/2} \|u\|_{Y(I \times \mathbf{R}^4)}.
\end{aligned} \tag{4.20}$$

Therefore, combining (4.10), (4.11), (4.14), (4.15), (4.19), and (4.20),

$$\|u\|_{Y(I \times \mathbf{R}^4)} \lesssim c(\eta)^{-1} + \eta \|u\|_{Y(I \times \mathbf{R}^4)}. \tag{4.21}$$

Choosing  $\eta > 0$  sufficiently small and then  $c(\eta) > 0$  sufficiently small, the proof of theorem 4.1 is complete.  $\square$

Now, armed with the long time Strichartz estimate we can rule out the rapid frequency cascade scenario.

**Theorem 4.2** *If  $u$  is an almost periodic solution to (1.1) on  $\mathbf{R}$ ,  $\int N(t)^{-2} dt = K < \infty$ , then  $u \equiv 0$ .*

*Proof:* Let  $k_0$  be the integer closest to  $k$ , where  $2^k = K^{-1/4}$ . Choose  $j \leq k_0$ . Then

$$\|\nabla P_{\leq j} u(t)\|_{L_x^2(\mathbf{R}^4)} \lesssim \|\nabla P_{\leq j} u(-T)\|_{L_x^2(\mathbf{R}^4)} + \|\nabla P_{\leq j} \int_{-T}^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau\|_{L_x^2(\mathbf{R}^4)}. \quad (4.22)$$

Also choose  $j < j_0(\eta)$  so that

$$\|P_{\leq j} u(t)\|_{L_t^\infty \dot{H}_x^1(\mathbf{R} \times \mathbf{R}^4)} \leq \eta. \quad (4.23)$$

Then

$$\|\nabla F(u_{\leq j})\|_{L_t^2 L_x^{4/3}([-T, T] \times \mathbf{R}^4)} \lesssim \eta^2 \|\nabla u_{\leq j}\|_{L_t^2 L_x^4([-T, T] \times \mathbf{R}^4)}. \quad (4.24)$$

Next, by Sobolev embedding,

$$\begin{aligned} & \|\nabla P_{\leq j} O(u_{\leq j}^2 u_{> j})\|_{L_t^2 L_x^{4/3}([-T, T] \times \mathbf{R}^4)} \\ & \lesssim 2^j \|u_{j < \cdot \leq k_0}\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^4)} \|\nabla u_{\leq j}\|_{L_t^2 L_x^4([-T, T] \times \mathbf{R}^4)} \|\nabla u_{\leq j}\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^4)}. \end{aligned} \quad (4.25)$$

Finally, by Bernstein's inequality and Sobolev embedding,

$$\begin{aligned} & \|\nabla P_{\leq j} O(u_{j < \cdot < k_0}^2 u)\|_{L_t^2 L_x^{4/3}([-T, T] \times \mathbf{R}^4)} \\ & \lesssim 2^{2j} \sum_{j \leq k_1 \leq k_2 \leq k_0} \|P_{k_1} u\|_{L_t^2 L_x^4([-T, T] \times \mathbf{R}^4)} \|P_{k_2} u\|_{L_t^\infty L_x^2([-T, T] \times \mathbf{R}^4)} \|u\|_{L_t^\infty L_x^4([-T, T] \times \mathbf{R}^4)}, \end{aligned} \quad (4.26)$$

and by theorem 4.1 and Sobolev embedding,

$$\|\nabla P_{\leq j} O(u_{\geq k_0}^3)\|_{L_t^2 L_x^{4/3}([-T, T] \times \mathbf{R}^4)} \lesssim 2^{2j} \|u_{\geq k_0}\|_{L_t^6 L_x^3([-T, T] \times \mathbf{R}^4)}^3 \lesssim 2^{2j} K^{1/2}, \quad (4.27)$$

and

$$\begin{aligned} & \|\nabla P_{\leq j} O(u_{\geq k_0}^2 u_{\leq k_0})\|_{L_t^2 L_x^{4/3}([-T, T] \times \mathbf{R}^4)} \\ & \lesssim 2^{3j/2} \|u_{\leq k_0}\|_{L_{t,x}^6([-T, T] \times \mathbf{R}^4)} \|u_{> k_0}\|_{L_t^6 L_x^3([-T, T] \times \mathbf{R}^4)} \lesssim 2^{3j/2} K^{1/2}. \end{aligned} \quad (4.28)$$

Therefore, by (4.22) - (4.28), by (3.8) and (4.24), for any  $T$ ,

$$\begin{aligned}
& \|\nabla P_{\leq j} u(t)\|_{L_t^2 L_x^4([-T, T] \times \mathbf{R}^4)} \\
& \lesssim \|\nabla P_{\leq j} u(-T)\|_{L_x^2(\mathbf{R}^4)} + \eta^2 \|\nabla P_{\leq j} u\|_{L_t^2 L_x^4([-T, T] \times \mathbf{R}^4)} \\
& + \sum_{j \leq l \leq k_0} 2^{2(j-l)} \|\nabla P_{\leq l} u(t)\|_{L_t^2 L_x^4([-T, T] \times \mathbf{R}^4)} + 2^{3j/2} K^{1/2}.
\end{aligned} \tag{4.29}$$

By theorem 4.1, for  $l \leq k_0$ ,

$$\|\nabla P_l u\|_{L_t^2 L_x^4([-T, T] \times \mathbf{R}^4)} \lesssim 1, \tag{4.30}$$

uniformly in  $T$ . Also, since  $\int_{\mathbf{R}} N(t)^{-2} dt = K < \infty$ ,  $N(-T) \nearrow +\infty$  as  $T \nearrow +\infty$ , so

$$\inf_T \|\nabla P_{\leq j} u(-T)\|_{L_x^2(\mathbf{R}^4)} = 0. \tag{4.31}$$

Therefore, by induction, starting with (4.30) for  $j_0(\eta) \leq l \leq k_0$ , (4.29) implies

$$\|\nabla P_{\leq j} u(t)\|_{L_t^2 L_x^4(\mathbf{R} \times \mathbf{R}^4)} \lesssim K^{1/2} 2^{3j/2}. \tag{4.32}$$

Also since

$$\begin{aligned}
\|\nabla P_{\leq j} u(t)\|_{L_t^\infty L_x^2([-T, T] \times \mathbf{R}^4)} & \lesssim \|\nabla P_{\leq j} u(-T)\|_{L_x^2(\mathbf{R}^4)} + \eta^2 \|\nabla P_{\leq j} u\|_{L_t^2 L_x^4([-T, T] \times \mathbf{R}^4)} \\
& + \sum_{j \leq l \leq k_0} 2^{2(j-l)} \|\nabla P_{\leq l} u(t)\|_{L_t^2 L_x^4([-T, T] \times \mathbf{R}^4)} + 2^{3j/2} K^{1/2},
\end{aligned} \tag{4.33}$$

$$\|\nabla P_{\leq j} u(t)\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^4)} \lesssim K^{1/2} 2^{3j/2}. \tag{4.34}$$

In particular this implies

$$\|u(t)\|_{H_x^{-1/4}(\mathbf{R}^4)} \lesssim K^{5/12}. \tag{4.35}$$

Then by Bernstein's inequality, interpolation, (3.8), and (4.35),

$$\begin{aligned}
\|u(t)\|_{L_x^2(\mathbf{R}^4)} & \lesssim \|P_{\leq \frac{1}{C(\eta)} N(t)} u(t)\|_{H_x^{-1/4}(\mathbf{R}^4)}^{4/5} \|P_{\leq \frac{1}{C(\eta)} N(t)} u(t)\|_{\dot{H}_x^1(\mathbf{R}^4)}^{1/5} \\
& + \|P_{\geq \frac{1}{C(\eta)} N(t)} u(t)\|_{L_x^2(\mathbf{R}^4)} \lesssim K^{2/3} \eta^{1/5} + \frac{C(\eta)}{N(t)}.
\end{aligned} \tag{4.36}$$

Then  $\int_{\mathbf{R}} N(t)^{-2} dt = K$  combined with (3.9) implies that  $N(t) \nearrow +\infty$  as  $t \nearrow \infty$ , so we can choose  $\eta(t) \searrow 0$ , possibly very slowly, such that

$$\|u(t)\|_{L_x^2(\mathbf{R}^4)} \rightarrow 0. \tag{4.37}$$

This implies  $u \equiv 0$ .  $\square$

## 5 Soliton

Now we turn to the case when  $\int_{\mathbf{R}} N(t)^{-2} dt = \infty$ . We begin by excluding the soliton, the case when  $N(t) = 1$  for all  $t \in \mathbf{R}$ . To do this we utilize an interaction Morawetz estimate. As in [13] and [30] the interaction Morawetz estimate utilizes an integral of mass estimate.

**Lemma 5.1** *Suppose that  $\psi \in C_0^\infty(\mathbf{R}^4)$ ,  $\psi = 1$  for  $|x| \leq 1$ , and  $\psi$  is supported on  $|x| \leq 2$ . For any  $1 \leq R \leq K^{1/5}$ , where  $\int_I N(t)^{-2} dt = \int_I 1 dt = K$ ,*

$$\int_I \int \int |u(t, y)|^2 \psi\left(\frac{x-y}{R}\right) [|\nabla u(t, x)|^2 + |u(t, x)|^4] dx dy dt \lesssim K \ln(R). \quad (5.1)$$

*Proof:* This is proved using the double Duhamel trick. See [33] for a similar result. Both here and in [33] we have a logarithmic - type failure. Suppose  $I = [t_-, t_+]$ .

Let  $P_h = P_{\geq K^{-1/4}}$ ,  $P_h + P_l = 1$ . By Duhamel's principle,

$$\begin{aligned} u_h(t) &= e^{i(t-t_-)\Delta} u_h(t_-) + \int_{t_-}^t e^{i(t-s_-)\Delta} P_h O(u_l u^2) ds_- + \int_{t_-}^t e^{i(t-s_-)\Delta} P_h F(u_h) ds_- \\ &= e^{i(t-t_+)\Delta} u_h(t_+) + \int_{t_+}^t e^{i(t-s_+)\Delta} P_h O(u_l u^2) ds_+ + \int_{t_+}^t e^{i(t-s_+)\Delta} P_h F(u_h) ds_+. \end{aligned} \quad (5.2)$$

Now for a fixed  $x \in \mathbf{R}^4$  define the inner product

$$\langle f, g \rangle_x = \int \psi\left(\frac{x-y}{R}\right) f(y) \overline{g(y)} dy. \quad (5.3)$$

Now use (2.27), (2.28), and (2.29). Let

$$\begin{aligned} A &= e^{i(t-t_-)\Delta} u_h(t_-) - i \int_{t_-}^t e^{i(t-s_-)\Delta} O(u_l u^2) ds_- \\ &\quad - i \int_{t_-}^{t-R^2} e^{i(t-s_-)\Delta} F(u_h) ds_- - i \int_{t-1}^t e^{i(t-s_-)\Delta} F(u_h) ds_-, \\ A' &= e^{i(t-t_+)\Delta} u_h(t_+) - i \int_{t_+}^t e^{i(t-s_+)\Delta} O(u_l u^2) ds_+ \\ &\quad - i \int_{t_+}^{t+R^2} e^{i(t-s_+)\Delta} F(u_h) ds_+ - i \int_{t+1}^t e^{i(t-s_+)\Delta} F(u_h) ds_+, \end{aligned} \quad (5.4)$$

and

$$\begin{aligned}
B &= -i \int_{t-R^2}^{t-1} e^{i(t-s_-)\Delta} F(u_h) ds_-, \\
B' &= -i \int_{t+R^2}^{t+1} e^{i(t-s_+)\Delta} F(u_h) ds_+.
\end{aligned} \tag{5.5}$$

By (4.13), (4.18), Strichartz estimates, Bernstein's inequality, and Hölder's inequality,

$$\int_I \int \psi\left(\frac{x-y}{R}\right) [|e^{i(t-t_-)\Delta} u(t_-)(y)|^2 + |\int_{t_-}^t e^{i(t-s_-)\Delta} O(u_l u^2)(y) ds_-|^2] dy dt \lesssim R^2 K^{1/2}. \tag{5.6}$$

By (2.18) and Hölder's inequality,

$$\begin{aligned}
& \int_{t_-}^{t-R^2} \int \psi\left(\frac{x-y}{R}\right) |e^{i(t-s_-)\Delta} F(u_h)(s_-)(y)|^2 dy \\
& \lesssim R^4 \left( \int_{t-s_- > R^2} \frac{1}{(t-s_-)^2} \|F(u_h)(s_-)\|_{L_x^1} ds_- \right)^2 \\
& \lesssim \left( \sum_{j \geq 0} \frac{1}{2^j R^2} \int_{|t-s_-| \sim 2^j R^2} \|F(u_h)(s_-)\|_{L_x^1(\mathbf{R}^4)} ds_- \right)^2 \lesssim (\mathcal{M}(\|F(s)\|_{L_x^1(\mathbf{R}^4)})(t))^2.
\end{aligned} \tag{5.7}$$

Finally, by Hölder's inequality in time and Sobolev embedding,

$$\|u_h^3\|_{L_t^2 L_x^{4/3}([t-1, t] \times \mathbf{R}^2)} \lesssim 1, \tag{5.8}$$

so by theorem 2.5 and (5.6) - (5.8),

$$|A|^2 \lesssim 1 + \mathcal{M}(\|u_h\|_{L_x^3(\mathbf{R}^4)}^3)(t)^2 + a(t)^2 R^2, \tag{5.9}$$

where  $\int a(t)^2 dt \lesssim K^{1/2}$ . By an identical calculation

$$|A'|^2 \lesssim 1 + \mathcal{M}(\|u_h\|_{L_x^3(\mathbf{R}^4)}^3)(t)^2 + a(t)^2 R^2. \tag{5.10}$$

To compute  $\langle B, B' \rangle_x$ , we compute the kernel of  $e^{i(t-s_-)\Delta} \psi(\frac{x-y}{R}) e^{i(s_+-t)\Delta}$ . Since  $t$  is fixed, to simplify notation let  $x = 0$ ,  $s = s_+ - t$  and  $t = t - s_-$ . Then the kernel of  $e^{it\Delta} \psi(\frac{y}{R}) e^{is\Delta}$  is given by

$$K(s, t; y, z) = \frac{C}{s^2 t^2} \int e^{-i \frac{|w-y|^2}{4t}} \psi\left(\frac{w}{R}\right) e^{-i \frac{|w-z|^2}{4s}} dw. \tag{5.11}$$

Now let  $q(s, t, y, z) = \frac{sy+zt}{s+t} \cdot \frac{(s+t)^{1/2}}{(ts)^{1/2}}$ . After making a change of variables in  $w$ ,

$$|K(s, t; y, z)| = \frac{C}{(s+t)^2} \int e^{-i|w-q(s,t,y,z)|^2} \psi\left(\frac{w}{R} \cdot \frac{(st)^{1/2}}{(s+t)^{1/2}}\right) dy. \tag{5.12}$$

When  $R \cdot \frac{(s+t)^{1/2}}{(st)^{1/2}} \leq 1$ , Hölder's inequality implies that  $|K(s, t; y, z)| \lesssim \frac{1}{(t+s)^2}$ .  
For  $R_0 = R \cdot \frac{(s+t)^{1/2}}{(st)^{1/2}} > 1$ , stationary phase calculations imply that for  $\chi \in C_0^\infty$ ,  $\chi = 1$  on  $|x| \leq 1$ , for any  $N$ ,

$$\int e^{-i|w-q|^2} (1-\chi)(w-q) \psi\left(\frac{w}{R_0}\right) dw = \int \left( \left( \frac{i(w-q) \cdot \nabla}{|w-q|^2} \right)^N e^{-i|w-q|^2} \right) (1-\chi)(w-q) \psi\left(\frac{w}{R_0}\right) dw. \quad (5.13)$$

Integrating by parts, for  $N = 5$ ,

$$|K(s, t; x, z)| \lesssim \frac{1}{(t+s)^2} \int_{|y|>1} \frac{1}{|y|^5} dy \lesssim \frac{1}{(t+s)^2}. \quad (5.14)$$

Therefore,

$$\begin{aligned} & \int_{1 < t-s_- < R^2} \int_{1 < s_+ - t < R^2} \langle e^{i(t-s_-)\Delta} F(u_h)(s_-), e^{i(t-s_+)\Delta} F(u_h)(s_+) \rangle_x ds_- ds_+ \\ & \lesssim \int_{1 < t-s_- < R^2} \int_{1 < s_+ - t < R^2} \frac{1}{(s_+ - s_-)^2} \|F(s_-)\|_{L_x^1} \|F(s_+)\|_{L_x^1} ds_- ds_+ \\ & \lesssim \sum_{0 \leq j \leq k \leq \ln(R)} 2^{-2k} \left( \int_{t-s_+ \sim 2^k} \|F(u_h)(s_+)\|_{L_x^1(\mathbf{R}^4)} ds_+ \right) \\ & \quad \times \left( \int_{t-s_- \sim 2^j} \|F(u_h)(s_-)\|_{L_x^1(\mathbf{R}^4)} ds_- \right) \lesssim \ln(R) \mathcal{M}(\|F(u_h)\|_{L_x^1(\mathbf{R}^4)})(t)^2. \end{aligned} \quad (5.15)$$

Therefore,

$$\langle u_h(t), u_h(t) \rangle_x \lesssim a(t)^2 R^2 + (\ln(R) + 1) \mathcal{M}(\|F(u_h)\|_{L_x^1(\mathbf{R}^4)})(t)^2 + 1, \quad (5.16)$$

and since  $\int [|\nabla u(t, x)|^2 + |u(t, x)|^4] dx < \|W\|_{H^1}^2 + \|W\|_{L_x^4}^4$ ,

$$\int_I \int \int \psi\left(\frac{x-y}{R}\right) |u_h(t, y)|^2 [|\nabla u(t, x)|^2 + |u(t, x)|^4] dx dy dt \lesssim (\ln(R) + 1) K. \quad (5.17)$$

Now notice that by the Sobolev embedding and theorem 4.1,

$$\|\nabla u_t^2\|_{L_t^{3/2} L_x^{12/5}(I \times \mathbf{R}^4)} \lesssim \|\nabla u_t\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)}^{4/3} \|u_t\|_{L_t^\infty L_x^4(I \times \mathbf{R}^4)}^{2/3} \lesssim 1, \quad (5.18)$$

so

$$\int_I \int \int |u_t(t, y)|^2 \psi\left(\frac{x-y}{R}\right) [|\nabla u(t, x)|^2 + |u(t, x)|^4] dx dy \lesssim K^{1/3} R^{10/3}. \quad (5.19)$$

Therefore the proof of lemma 5.1 is complete.  $\square$

Now we are ready to exclude the soliton scenario.

**Theorem 5.2** *Suppose  $u$  is an almost periodic solution to (1.1) with  $N(t) \equiv 1$  on  $\mathbf{R}$  and with  $\dot{H}^1$  norm below the threshold. Then  $u \equiv 0$ .*

We prove this by constructing an interaction Morawetz estimate suited to the focusing problem. This Morawetz estimate is in the same vein as [17] and [18].

*Proof:* Define a function  $\psi \in C_0^\infty(\mathbf{R})$ ,  $\psi$  even,  $\psi = 1$  for  $|x| \leq 1$  and  $\psi = 0$  for  $|x| > 2$ . Then let

$$\phi(x-y) = \int \psi^2(x-s)\psi^2(y-s)ds. \quad (5.20)$$

Notice that  $\phi$  is supported on  $|x| \leq 4$ . Then define the interaction Morawetz potential

$$M_R(t) = \int |u(t, y)|^2 \phi\left(\frac{x-y}{R}\right) (x-y)_j \operatorname{Im}[\bar{u} \partial_j u](t, x) dx dy. \quad (5.21)$$

By Hölder's inequality and Sobolev embedding,

$$\sup_{t \in I} |M_R(t)| \lesssim R^4. \quad (5.22)$$

By direct calculation,

$$\frac{d}{dt} M_R(t) = 2 \int |u(t, y)|^2 \phi\left(\frac{x-y}{R}\right) [|\nabla u(t, x)|^2 - |u(t, x)|^4] dx dy \quad (5.23)$$

$$- 2 \int \operatorname{Im}[\bar{u} \partial_j u](t, y) \phi\left(\frac{x-y}{R}\right) \operatorname{Im}[\bar{u} \partial_j u](t, x) dx dy \quad (5.24)$$

$$+ 2 \int |u(t, y)|^2 \phi'\left(\frac{x-y}{R}\right) \frac{(x-y)_j (x-y)_k}{|x-y|R} [\operatorname{Re}(\partial_j \bar{u} \partial_k u)(t, x) - \frac{1}{4} \delta_{jk} |u(t, x)|^4] dx dy \quad (5.25)$$

$$- 2 \int \operatorname{Im}[\bar{u} \partial_k u](t, y) \phi'\left(\frac{x-y}{R}\right) \frac{(x-y)_j (x-y)_k}{|x-y|R} \operatorname{Im}[\bar{u} \partial_j u](t, x) dx dy \quad (5.26)$$

$$- \frac{1}{2} \int |u(t, y)|^2 \Delta [4\phi\left(\frac{x-y}{R}\right) + \phi'\left(\frac{x-y}{R}\right) \frac{|x-y|}{R}] |u(t, x)|^2 dx dy. \quad (5.27)$$

First consider (5.25) + (5.26). Take  $R_0 = K^{1/5}$ . By the support of  $\phi(x)$ ,

$$\int_{1 \leq R \leq R_0} \frac{1}{R} |\phi'(\frac{x-y}{R}) \frac{(x-y)_j(x-y)_k}{|x-y|R}| dR \lesssim 1, \quad (5.28)$$

and is supported on  $|x-y| \lesssim R_0$ . Therefore, by lemma 5.1 and the Cauchy - Schwartz inequality,

$$\int_I (5.25) + (5.26) dt \lesssim K \ln(R_0). \quad (5.29)$$

Next take (5.27). Because  $\phi(x-y)$  is supported on  $|x-y| \leq 4$ ,

$$\int_{1 \leq R \leq R_0} \frac{1}{R} |\Delta[4\phi(\frac{x-y}{R}) + \phi'(\frac{x-y}{R}) \frac{|x-y|}{R}]| dR \lesssim \int_{1 \leq R \leq R_0} \frac{1}{R^3} \phi(\frac{x-y}{2R}) dR \lesssim \frac{1}{1+|x-y|^2}, \quad (5.30)$$

and is also supported on  $|x-y| \lesssim R_0$ . Take the mollifier  $\chi \in C_0^\infty(\mathbf{R}^4)$ ,  $\int \chi(x) = 1$ ,  $\chi \geq 0$ , and  $\chi$  supported on  $|x| \leq \frac{1}{4}$ . Then

$$u_h(t, x) = u_h(t, x) - \frac{1}{R^4} \int \chi(\frac{x-y}{R}) u_h(t, y) dy + \frac{1}{R^4} \int \chi(\frac{x-y}{R}) u_h(t, y) dy. \quad (5.31)$$

By the fundamental theorem of calculus,

$$\begin{aligned} u_h(t, x) - \frac{1}{R^4} \int \chi(\frac{x-y}{R}) u_h(t, y) dy &= \frac{1}{R^4} \int \chi(\frac{x-y}{R}) [u_h(t, x) - u_h(t, y)] dy \\ &= \frac{1}{R^4} \int \chi(\frac{z}{R}) \int_0^1 [\nabla u_h(t, x + sz)] \cdot z ds dz. \end{aligned} \quad (5.32)$$

Then by (5.32) and lemma 5.1,

$$\begin{aligned} \int_{1 \leq R \leq R_0} \frac{1}{R^3} \int \int_{|x-y| \sim R} |u_h(t, y)|^2 |u_h(t, x) - \frac{1}{R^4} \int \chi(\frac{x-w}{R}) u_h(t, w) dw|^2 dx dy dt dR \\ \lesssim \int_I \int_{|x-y| \lesssim R_0} |u_h(t, y)|^2 |\nabla u_h(t, x)|^2 dx dy dt dR \lesssim K \ln(R_0). \end{aligned} \quad (5.33)$$

By theorem 4.1, since  $\hat{\chi}(\xi)$  is rapidly decreasing for  $|\xi| \geq 1$  and Hölder's inequality,

$$\begin{aligned} \int_{1 \leq R \leq R_0} \frac{1}{R^3} \int \int_{|x-y| \sim R} |\frac{1}{R^4} \int \chi(\frac{y-w}{R}) u(t, w) dw|^2 |u_h(t, x)|^2 dx dy dt dR \\ \lesssim \int_{1 \leq R \leq R_0} \left( \sum_{K^{-1/4} \leq M_1 \leq M_2} \frac{1}{1 + M_1^4 R^4} \|P_{M_1} u\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)} \frac{1}{1 + M_2^4 R^4} \right. \\ \left. \times \|P_{M_2} u\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)} \|P_{>M_2} u\|_{L_t^\infty L_x^{8/3}(I \times \mathbf{R}^4)}^2 \right) dR \lesssim K \ln(R_0). \end{aligned} \quad (5.34)$$



**Remark:** In fact by Bernstein's inequality, (3.8), and  $N(t) \geq 1$ ,

$$\|u_{>M_2}\|_{L_t^\infty L_x^{8/3}(I \times \mathbf{R}^4)} \lesssim o\left(\frac{1}{M_2}\right), \quad (5.35)$$

where  $M_2 o(\frac{1}{M_2}) \rightarrow 0$  as  $M_2 \searrow 0$ . Therefore,

$$(5.34) \lesssim Ko(\ln(R_0)). \quad (5.36)$$

(5.36) will be crucial later. Finally, by theorem 3.1, theorem 4.1, and Sobolev embedding,

$$\|u_t^2\|_{L_t^2 L_x^3(I \times \mathbf{R}^4)} \lesssim \|\nabla u_t\|_{L_t^2 L_x^4(I \times \mathbf{R}^4)} \|u\|_{L_t^\infty L_x^3(I \times \mathbf{R}^4)} \lesssim 1, \quad (5.37)$$

so by Hölder's inequality

$$\int_{1 \leq R \leq R_0} \frac{1}{R^3} \int_I \int \int_{|x-y| \sim R} |u_t(t, x)|^2 |u_t(t, y)|^2 dx dy dt dR \lesssim K^{1/2} \int_{1 \leq R \leq R_0} R dR \lesssim K^{1/2} R_0^2. \quad (5.38)$$

Now consider (5.23) and (5.24). Recall that

$$\phi\left(\frac{x-y}{R}\right) = \int \psi^2\left(\frac{x}{R} - s\right) \psi^2\left(\frac{y}{R} - s\right) ds. \quad (5.39)$$

For each  $s, t$  there exists a  $\xi(s, t)$  such that

$$\int \psi^2\left(\frac{x}{R} - s\right) \text{Im}[\bar{u} \nabla e^{ix \cdot \xi(s, t)} u](t, x) dx = 0. \quad (5.40)$$

Moreover, the quantity

$$\int \psi^2\left(\frac{x}{R} - s\right) \psi^2\left(\frac{y}{R} - s\right) [|\nabla u(t, x)|^2 |u(t, y)|^2 - \text{Im}[\bar{u} \nabla u](t, x) \text{Im}[\bar{u} \nabla u](t, y)] dx dy \quad (5.41)$$

is invariant under the Galilean transformation  $u \mapsto e^{-ix \cdot \xi(s, t)} u$ . Therefore, for each  $t$  we can take a Galilean transform on each square  $\psi^2(x - s) \psi^2(y - s)$  separately to rid ourselves of the momentum squared term. Now for a fixed  $t, s$ ,

$$\begin{aligned} \int \psi^2\left(\frac{x}{R} - s\right) [|\nabla e^{-ix \cdot \xi(s, t)} u(t, x)|^2 - |u(t, x)|^4] dx &= \int |u(t, x)|^2 (\psi\left(\frac{x}{R} - s\right) \Delta \psi\left(\frac{x}{R} - s\right)) dx \\ &+ \int |\nabla (\psi\left(\frac{x}{R} - s\right) e^{-ix \cdot \xi(s, t)} u(t, x))|^2 dx - |\psi\left(\frac{x}{R} - s\right) u(t, x)|^2 |u(t, x)|^2 dx. \end{aligned} \quad (5.42)$$

By lemma 1.4,  $\|u\|_{\dot{H}^1} < (1 - \bar{\delta}) \|W\|_{\dot{H}^1}$ , so by (1.19),

$$\|u\|_{L^4(\mathbf{R}^4)} \leq (1 - \bar{\delta}) \|W\|_{L^4(\mathbf{R}^4)}, \quad (5.43)$$

and therefore since  $C_4 = \frac{1}{\|W\|_{L^4(\mathbb{R}^4)}}$ ,

$$\begin{aligned} \int |\nabla(\psi(\frac{x}{R} - s)e^{-ix \cdot \xi(s,t)}u(t, x))|^2 dx - |\psi(\frac{x}{R} - s)e^{-ix \cdot \xi(s,t)}u(t, x)|^2 |u(t, x)|^2 dx \\ \gtrsim \bar{\delta} \int \psi(\frac{x}{R} - s)^2 |u(t, x)|^4 dx. \end{aligned} \quad (5.44)$$

It remains to calculate

$$\int_{1 \leq R \leq R_0} \frac{1}{R} \int_I \int \int |\psi(\frac{x}{R} - s) \Delta \psi(\frac{x}{R} - s)| |u(t, x)|^2 |u(t, y)|^2 \psi^2(\frac{y}{R} - s) dx dy ds dt dR. \quad (5.45)$$

Now if  $|\frac{x}{R} - s| \sim 1$  and  $|\frac{y}{R} - s| \lesssim 1$ ,  $|x - y| \lesssim R$ . Moreover,  $|\Delta \psi(\frac{x}{R} - s)| \lesssim \frac{1}{R^2}$ . Now,

$$\begin{aligned} \int_{1 \leq R \leq R_0} \frac{1}{R^3} \int \int_{|x-s| \sim R} \psi^2(y-s) |u(t, x)|^2 |u(t, y)|^2 dx dy ds dt dR \\ \lesssim \int_I \int_{|x-y| \lesssim R_0} \frac{1}{1+|x-y|^2} |u(t, y)|^2 |u(t, x)|^2 dx dy dt. \end{aligned} \quad (5.46)$$

Then by (5.33), (5.34), and (5.38),

$$(5.46) \lesssim K \ln(R_0) + K^{1/2} R_0^2. \quad (5.47)$$

Therefore, by (5.44), the fundamental theorem of calculus, (5.22), (5.29), (5.33), (5.34), (5.38), and (5.47),

$$\begin{aligned} \bar{\delta} \ln(R_0) \int_I \int_{|x-y| \leq R_0^{1/2}} |u(t, x)|^4 |u(t, y)|^2 dx dy dt - O(K \ln(R_0)) - O(K^{1/2} R_0^2) \\ \lesssim \int_{1 \leq R \leq R_0} \int_I \frac{1}{R} \frac{d}{dt} M_R(t) dt dR \lesssim R_0^4. \end{aligned} \quad (5.48)$$

Therefore, if  $R_0 \leq K^{1/5}$ ,

$$\bar{\delta} \int_I \int_{|x-y| \leq R_0^{1/2}} |u(t, x)|^4 |u(t, y)|^2 dx dt \lesssim K. \quad (5.49)$$

Now notice that (5.49) represents a logarithmic improvement over the result of lemma 5.1 for the term involving  $|u(t, x)|^4$ . Moreover, because  $u(t)$  lies in a compact subset of  $\dot{H}^1$  modulo scaling symmetries, we can make a point set topology argument to prove that  $\|u(t)\|_{L_x^4}$  is uniformly bounded below. Then by (3.8), (4.2),

$$\int_{|x-x(t)| \leq \frac{C(\eta)}{N(t)}} |\nabla u(t, x)|^2 dx \sim \int_{|x-x(t)| \leq \frac{C(\eta)}{N(t)}} |u(t, x)|^4 dx. \quad (5.50)$$

This gives a type of inverse Sobolev embedding (see [13]) that is useful to control the kinetic energy term. By (5.50),

$$\begin{aligned} \int_{1 \leq R \leq R_0^{1/2}} \frac{1}{R} \int_I \int \int_{|y-x(t)| \leq R-C(\eta)} |u(t, y)|^2 \psi\left(\frac{x-y}{R}\right) \frac{|x-y|}{R} |\nabla u(t, x)|^2 dx dy dt dR \\ \lesssim \int_I \int \int \psi\left(\frac{x-y}{R_0}\right) |u(t, y)|^2 |u(t, x)|^4 dx dy dt. \end{aligned} \quad (5.51)$$

If  $|x(t) - y| > R + C(\eta)$ , then by lemma 5.1 and (3.8),

$$\int_{1 \leq R \leq R_0} \frac{1}{R} \int_I \int \int_{|x(t)-y| > R+C(\eta)} |u(t, y)|^2 \psi\left(\frac{x-y}{R}\right) \frac{|x-y|}{R} |\nabla u(t, x)|^2 dx dy dt dR \lesssim \eta K \ln(R_0). \quad (5.52)$$

Finally, by lemma 5.1,

$$\begin{aligned} \int_{1 \leq R \leq R_0^{1/2}} \frac{1}{R} \int_I \int \int_{R-C(\eta) < |x(t)-y| < R+C(\eta)} \int |\nabla u(t, x)|^2 \psi\left(\frac{x-y}{R}\right) \frac{|x-y|}{R} |u(t, y)|^2 dx dy dt dR \\ \lesssim \int_I \int \int_{|x-y| \leq 2R_0} \frac{C(\eta)}{C(\eta) + |x-y|} |\nabla u(t, x)|^2 |u(t, y)|^2 dx dy dt \lesssim K \ln(C(\eta)). \end{aligned} \quad (5.53)$$

Therefore,

$$\begin{aligned} \int_I \int_{|x-y| \leq \frac{R_0^{1/2}}{2}} |\nabla u(t, x)|^2 |u(t, y)|^2 dx dy dt \\ \lesssim \int_{1 \leq R \leq R_0^{1/2}} \frac{1}{R} \int \int |u(t, y)|^2 |\nabla u(t, x)|^2 \psi\left(\frac{x-y}{R}\right) \frac{|x-y|}{R} dx dy dt dR \\ \lesssim \eta K \ln(R_0) + K \ln(C(\eta)). \end{aligned} \quad (5.54)$$

Therefore,

$$\int_I (5.25) + (5.26) dt \lesssim \eta K \ln(R_0) + K \ln(C(\eta)) + K^{1/2} R_0^2. \quad (5.55)$$

Next, by (5.33), (5.34), and (5.38),

$$\int_{1 \leq R \leq R_0^{1/2}} \frac{1}{R} \int_I (5.27) dt dR \lesssim o(\ln(R_0))K + \eta K \ln(R_0) + K \ln(C(\eta)). \quad (5.56)$$

Then by (5.47) and the fundamental theorem of calculus

$$\begin{aligned} \int_I \int_{|x-y| \leq R_0^{1/4}} \ln(R_0) |u(t, x)|^4 |u(t, y)|^2 dx dy dt \\ \lesssim o(\ln(R_0))K + \eta K \ln(R_0) + K \ln(C(\eta)) + R_0^2 K^{1/2} + R_0^4. \end{aligned} \quad (5.57)$$

However, by (4.1) this implies that either there exists a sequence  $t_n \in \mathbf{R}$  such that  $R_{0,n} \nearrow \infty$  and either

$$\int_{|x-x(t_n)| \leq R_{0,n}^{1/4}} |u(t_n, x)|^2 dx \rightarrow 0, \quad (5.58)$$

or

$$\int_{|x-x(t_n)| \leq R_{0,n}^{1/4}} |u(t_n, x)|^4 dx \rightarrow 0. \quad (5.59)$$

In either case, this implies that  $u \equiv 0$ .  $\square$

## 6 Variable $N(t)$

Now we turn to the case when  $N(t)$  is free to vary. In this case we may wish to try

$$M(t) = \int |u(t, y)|^2 \phi\left(\frac{(x-y)N(t)}{R}\right) (x-y)_j \operatorname{Im}[\bar{u} \partial_j u](t, x) dx dy. \quad (6.1)$$

Everything would then proceed exactly as in theorem 5.2, except that we have one additional term,

$$\int |u(t, y)|^2 \phi\left(\frac{(x-y)N(t)}{R}\right) \frac{|x-y|(x-y)_j}{R} N'(t) \operatorname{Im}[\bar{u} \partial_j u](t, x) dx dy. \quad (6.2)$$

Notice that by Hölder's inequality and  $\|u\|_{L_t^\infty L_x^4(I \times \mathbf{R}^4)} \lesssim 1$ ,  $(6.2) \lesssim R^4 \frac{N'(t)}{N(t)^5}$ .

In the case that  $\int \frac{N'(t)}{N(t)^5} dt \ll K$ , we would be done. So this would rule out not only the case when  $N(t) \equiv 1$ , but also the case when  $N(t)$  is a monotone function. However,  $N(t)$  may be highly oscillatory. In that case, it is useful to replace  $N(t)$  with  $\tilde{N}(t)$ , that satisfies the following conditions:

1.  $\tilde{N}(t) \gtrsim 1$ .
2.  $|\tilde{N}'(t)| \lesssim \tilde{N}(t)^3$ .
- 3.

$$\int_I \frac{1}{\tilde{N}(t)^2} dt \lesssim K, \quad (6.3)$$

and

- 4.
- $$\int_I \frac{|\tilde{N}'(t)|}{\tilde{N}(t)^5} dt \ll K. \quad (6.4)$$

$\tilde{N}(t)$  will be inductively defined, using a procedure very similar to the construction in [17]. We begin with  $\tilde{N}_0(t)$ , although to simplify notation we will simply write  $N_0(t)$ .

**Definition 6.1** *Let*

$$\frac{1}{N_0(t)} = \|u_h(t)\|_{L_x^3(\mathbf{R}^4)}^3. \quad (6.5)$$

**Lemma 6.1** *Possibly after modifying  $N_0(t)$  by some function  $\alpha(t)$ ,  $N_0(t) \mapsto \alpha(t)N_0(t)$ ,*

$$\epsilon < \alpha(t) < \frac{1}{\epsilon}, \quad (6.6)$$

$$1. \ N_0(t) \gtrsim 1.$$

$$2. \ |N'_0(t)| \lesssim N_0(t)^3,$$

and

$$3.$$

$$\int_I \frac{1}{N_0(t)^2} dt \lesssim K. \quad (6.7)$$

*Proof:*  $N_0(t) \gtrsim 1$  follows directly from theorem 3.1. Notice also that by (3.8) and interpolation  $N_0(t) \lesssim N(t)$ . Next, take  $t_0 \in \mathbf{R}$  and choose  $N_0 \sim N(t_0)$  such that by Bernstein's inequality,

$$\|P_{>N_0} u(t)\|_{L_t^\infty L_x^3(\mathbf{R}^4)} \leq 10^{-6} N(t_0)^{-1}. \quad (6.8)$$

By Sobolev embedding,

$$\begin{aligned} \frac{d}{dt} \left( \int |P_{\leq N_0} u_h(t, x)|^3 dx \right) &= \left( \int |P_{\leq N_0} u_h(t, x)| \operatorname{Re}((i\Delta P_{\leq N_0} u_h + iP_{\leq N_0} P_h F(u)) P_{\leq N_0} \bar{u}_h) dx \right) \\ &\lesssim \left( \int |\nabla P_{\leq N_0} u_h(t, x)|^2 |P_{\leq N_0} u_h(t, x)| dx + \int |P_{\leq N_0} u_h(t, x)|^2 |P_{\leq N_0} P_h F(u)(t, x)| dx \right) \lesssim N_0. \end{aligned} \quad (6.9)$$

Then for  $c > 0$  sufficiently small, for any  $|t - t_0| \leq cN(t_0)^{-2}$ ,

$$\|P_{\leq N_0} u_h(t)\|_{L_x^3(\mathbf{R}^4)} \gtrsim N(t_0)^{-1}. \quad (6.10)$$

Therefore, by Bernstein's inequality, for  $|t_0 - t_1| \leq cN(t_0)^{-2}$ ,

$$N_0(t_0) \sim N_0(t_1), \quad (6.11)$$

and thus  $|N'_0(t)| \lesssim N_0(t)^3$ , possibly after modifying  $N_0(t)$  by a constant.

Finally, by theorem 4.1, (6.7) holds.  $\square$

**Theorem 6.2** *If  $u$  is an almost periodic solution to (1.1) with  $\int_{\mathbf{R}} N(t)^{-2} dt = \infty$ , then  $u \equiv 0$ .*

*Proof:* Analogously to lemma 5.1 define the inner product

$$\langle f, g \rangle_x = \int \psi\left(\frac{(x-y)N_0(t)}{R}\right) f(y) \overline{g(y)} dy. \quad (6.12)$$

$$\int_{|t-s| > \frac{R^4}{N_0(t)^2}} \|e^{i(t-s)\Delta} F(u_h)(s) ds\|_{L_x^\infty} ds \lesssim \frac{N_0(t)^2}{R^2} \mathcal{M}(\|u_h\|_{L^3}^3)(t). \quad (6.13)$$

$$\left\| \int_{|t-s| < \frac{1}{N_0(t)^2}} e^{i(t-s)\Delta} F(u_h)(s) ds \right\|_{L^2(\mathbf{R}^4)} \lesssim \frac{1}{N_0(t)}. \quad (6.14)$$

Next, by (5.14),

$$\int_{\frac{-1}{N_0(t)^2} + t}^{\frac{-1}{N_0(t)^2} + t} \int_{\frac{-R^2}{N_0(t)^2} + t}^{\frac{R^2}{N_0(t)^2} + t} \langle e^{i(t-s_+)\Delta} F(u_h)(s_+), e^{i(t-s_-)\Delta} F(u_h)(s_-) \rangle_x ds_+ ds_- \lesssim \mathcal{M}(\|u_h\|_{L_x^3}^3)(t)^2. \quad (6.15)$$

Therefore by (5.6),  $N_0(t) \gtrsim 1$ , (6.7), (6.13), (6.14), and (6.15), and (5.18),

$$\begin{aligned} & \int_I \int |u(t, y)|^2 \psi\left(\frac{(x-y)N_0(t)}{R}\right) [|\nabla u(t, x)|^2 + |u(t, x)|^4] dx dy dt \\ & \lesssim K \ln(R) + K^{1/2} R^2 + \left( \int_I \frac{R^{10}}{N_0(t)^{10}} dt \right)^{1/3} \lesssim K \ln(R) + K^{1/2} R^2 + K^{1/3} R^{10/3}. \end{aligned} \quad (6.16)$$

Next, by a calculation similar to (5.33),

$$\begin{aligned} & \int_{1 \leq R \leq R_0} \frac{N_0(t)^2}{R^3} \int_I \int_{|x-y| \leq 2\frac{R}{N_0(t)}} |u_h(t, y)|^2 \\ & \times |u_h(t, x) - \frac{N_0(t)^4}{R^4} \int \chi\left(\frac{(x-w)N_0(t)}{R}\right) u_h(t, w) dw|^2 dx dy dt dR \\ & \lesssim \int_I \int_{|x-y| \leq 2\frac{R_0}{N_0(t)}} |\nabla u_h(t, x)|^2 |u_h(t, y)|^2 dx dy dt \lesssim \ln(R_0) K + R_0^2 K^{1/2}. \end{aligned} \quad (6.17)$$

Also,

$$\begin{aligned}
& \int_{1 \leq R \leq R_0} \frac{N_0(t)^2}{R^3} \int_I \int_{|x-y| \leq 2\frac{R}{N_0(t)}} \left| \frac{N_0(t)^4}{R^4} \int \chi\left(\frac{(y-w)N_0(t)}{R}\right) u_h(t, w) dw \right|^2 \\
& \quad \times \left| u_h(t, x) - \frac{N_0(t)^4}{R^4} \int \chi\left(\frac{(x-w)N_0(t)}{R}\right) u_h(t, w) dw \right|^2 dx dy dt dR \\
& \lesssim \int_I \int_{|x-y| \leq 2\frac{R_0}{N_0(t)}} |\nabla u_h(t, x)|^2 |u_h(t, y)|^2 dx dy dt \lesssim \ln(R_0)K + R_0^2 K^{1/2}.
\end{aligned} \tag{6.18}$$

Finally, by Hölder's inequality, the fact that  $\hat{\chi}$  is rapidly decreasing for  $|\xi| \geq 1$ , (3.8), and theorem 4.1,

$$\begin{aligned}
& \int_I \frac{N_0(t)^2}{R^3} \int_{|x-y| \leq 2\frac{R}{N_0(t)}} \left| \frac{N_0(t)^4}{R^4} \int \chi\left(\frac{N_0(t)(y-w)}{R}\right) u_h(t, w) dw \right|^2 \\
& \quad \times \left| \frac{N_0(t)^4}{R^4} \int \chi\left(\frac{(x-w)N_0(t)}{R}\right) u_h(t, w) dw \right|^2 dx dy dt \\
& \lesssim \int_I \sum_{K_1^{-1/4} \leq N_1 \leq N_2 \leq N_3 \leq N_4} \frac{RN_0(t)^2}{N_0(t)^4 + R^4 N_4^4} \|P_{N_1} u_h(t)\|_{L_x^\infty(\mathbf{R}^4)} \|P_{N_2} u_h(t)\|_{L_x^\infty(\mathbf{R}^4)} \\
& \quad \times \|P_{N_3} u_h(t)\|_{L_x^2(\mathbf{R}^4)} \|P_{N_4} u_h(t)\|_{L_x^2(\mathbf{R}^4)} dt \\
& \lesssim o\left(\frac{1}{R}\right) \sup_{M \geq K_1^{-1/4}} M^{-2} \|P_M u_h(t)\|_{L_x^\infty(\mathbf{R}^4)}^2 \| \nabla u_h \|^2_{L_t^\infty L_x^2(I \times \mathbf{R}^4)} \lesssim Ko\left(\frac{1}{R}\right).
\end{aligned} \tag{6.19}$$

$$\int_{1 \leq R \leq R_0} o\left(\frac{1}{R}\right) K dR \lesssim Ko(\ln(R_0)). \tag{6.20}$$

Finally, by Hölder's inequality, Sobolev embedding, theorem 4.1, theorem 3.1, and (5.37),

$$\begin{aligned}
& \int_{1 \leq R \leq R_0} \frac{N_0(t)^2}{R^3} \int_I \int_{|x-y| \lesssim \frac{R}{N_0(t)}} |u_l(t, y)|^2 |u_l(t, x)|^2 dx dy dt dR \\
& \lesssim \int_{1 \leq R \leq R_0} \left( \int_I \frac{R^2}{N_0(t)^4} dt \right)^{1/2} dR \lesssim R_0^2 K^{1/2}.
\end{aligned} \tag{6.21}$$

Therefore, by (6.17) - (6.21),

$$\int_{1 \leq R \leq R_0} \int_I \frac{N_0(t)^2}{R^3} \int_{|x-y| \leq 2\frac{R}{N_0(t)}} |u(t, y)|^2 |u(t, x)|^2 dx dy dt dR \lesssim K \ln(R_0) + K^{1/2} R_0^2. \tag{6.22}$$

Now we will define an interaction Morawetz estimate with  $N_m(t) \geq N_0(t)$ ,  $N_m(t)$  becoming progressively smoother in time after each iteration. Let

$$M_R(t) = \int_I \int \int \psi\left(\frac{(x-y)N_m(t)}{R}\right)(x-y)_j |u(t,y)|^2 \operatorname{Im}[\bar{u}\partial_j u](t,x) dx dy. \quad (6.23)$$

Then  $|M_R(t)| \lesssim \frac{R^4}{N_m(t)^4} \lesssim R^4$  and

$$\frac{d}{dt} M_R(t) = 2 \int \int \psi\left(\frac{(x-y)N_m(t)}{R}\right) |u(t,y)|^2 [|\nabla u(t,x)|^2 - |u(t,x)|^4] dx dy \quad (6.24)$$

$$- 2 \int \int \psi\left(\frac{(x-y)N_m(t)}{R}\right) \operatorname{Im}[\bar{u}\partial_j u](t,x) \operatorname{Im}[\bar{u}\partial_j u](t,y) dx dy \quad (6.25)$$

$$+ 2 \int \int \psi'\left(\frac{(x-y)N_m(t)}{R}\right) \frac{(x-y)_j(x-y)_k}{|x-y|R} |u(t,y)|^2 [Re(\partial_j \bar{u}\partial_k u)(t,x) - \frac{\delta_{jk}}{4} |u(t,x)|^4] dx dy \quad (6.26)$$

$$- 2 \int \int \operatorname{Im}[\bar{u}\partial_k u](t,y) \psi'\left(\frac{(x-y)N_m(t)}{R}\right) \frac{(x-y)_j(x-y)_k}{|x-y|R} \operatorname{Im}[\bar{u}\partial_j u](t,x) dx dy \quad (6.27)$$

$$+ \frac{1}{2} \int \int |u(t,y)|^2 \Delta [4\psi\left(\frac{(x-y)N_m(t)}{R}\right) + \psi'\left(\frac{(x-y)N_m(t)}{R}\right) \frac{|x-y|N_m(t)}{R}] |u(t,x)|^2 dx dy \quad (6.28)$$

$$+ \int \int \psi'\left(\frac{(x-y)N_m(t)}{R}\right) \frac{(x-y)_j |x-y|N'_m(t)}{R} |u(t,y)|^2 \operatorname{Im}[\bar{u}\partial_j u](t,x) dx dy. \quad (6.29)$$

Now by (5.42) – (5.44),

$$\begin{aligned} (6.24) + (6.25) &\geq \bar{\delta} \int \psi\left(\frac{(x-y)N_m(t)}{R}\right) |u(t,y)|^2 |u(t,x)|^4 dx dy \\ &\quad - \frac{CN_m(t)^2}{R^2} \int_{|x-y| \leq 2\frac{R}{N_m(t)}} |u(t,y)|^2 |u(t,x)|^2 dx dy. \end{aligned} \quad (6.30)$$

Therefore, by (6.16) – (6.21), for  $R_0 \leq K^{1/5}$ ,



$$\begin{aligned}
& \int_I \ln(R_0) \int \int_{|x-y| \leq \frac{R_0^{11/12}}{N_m(t)}} |u(t, y)|^2 |u(t, x)|^4 dx dy dt - K \ln(R_0) - K^{1/2} R_0^2 \\
& - \int_{1 \leq R \leq R_0} \frac{1}{R} \int_I \int \psi' \left( \frac{(x-y)N_m(t)}{R} \right) \frac{(x-y)_j |x-y| N'_m(t)}{R} \\
& \quad \times |u(t, y)|^2 \operatorname{Im}[\bar{u} \partial_j u](t, x) dx dy dt dR \\
& \gtrsim \int_I \int_{1 \leq R \leq R_0} \frac{1}{R} \frac{d}{dt} M_R(t) dt \lesssim R_0^4.
\end{aligned} \tag{6.31}$$

Now we apply a smoothing procedure to make  $N_m(t)$  much smoother than  $N_0(t)$ . See [17] for a similar procedure. Partition  $I$  into subintervals  $J_k$  such that  $\int_{J_k} N_0(t)^2 dt = c$  for some  $c \ll 1$ . Then let

$$N(J_k) = \sup\{2^j : j \in \mathbf{Z}, \quad 2^j \leq N(t) \quad \forall t \in J_k\}. \tag{6.32}$$

For  $c$  sufficiently small, if  $J_k$  and  $J_{k+1}$  are adjacent intervals then  $|N'_0(t)| \lesssim N_0^3(t)$  implies

$$\frac{N(J_k)}{N(J_{k+1})} = 1, \quad \frac{1}{2}, \quad \text{or } 2 \tag{6.33}$$

and  $|J_k| \sim N(J_k)^{-2}$ . Then choose  $N_1(t_k) = N(J_k)$ , where  $t_k$  is the midpoint of  $J_k$  and let  $N_1(t)$  be the linear interpolation between these midpoints. Then

$$\int_I \frac{|N'_1(t)|}{N_1(t)^5} dt \lesssim K. \tag{6.34}$$

Now we iteratively obtain  $N_{l+1}(t)$  from  $N_l(t)$  using the smoothing algorithm.

**Definition 6.2 (Smoothing algorithm)** *An interval  $J_k$  is called upward sloping if  $\frac{N(J_k)}{N(J_{k+1})} = \frac{1}{2}$ , downward sloping if  $\frac{N(J_k)}{N(J_{k+1})} = 2$ , and flat if  $\frac{N(J_k)}{N(J_{k+1})} = 1$ . We call  $J$  a valley if  $J = J_l \cup J_{l+1} \cup \dots \cup J_{l+m}$ ,  $J_l$  is downward sloping,  $J_{l+m}$  is upward sloping, and  $J_{l+1}, \dots, J_{l+m-1}$  are constant intervals. We call  $J$  a peak if  $J = J_l \cup J_{l+1} \cup \dots \cup J_{l+m}$ ,  $J_l$  is upward sloping,  $J_{l+m}$  is downward sloping, and  $J_{l+1}, \dots, J_{l+m-1}$  are constant intervals.*

**Remark:**  $N_1(t)$  is monotone in between consecutive peaks and valleys. Moreover, we cannot have two peaks without a valley in between, or two valleys without a peak in between.

Now if

$$J = J_l \cup \dots \cup J_{m+1}, \tag{6.35}$$

is a valley let  $N_2(t) = N_1(t_l) = N_j(t_l)$  for all  $t_l < t < t_{l+m}$ . Otherwise let  $N_2(t) = N_1(t)$ .

Likewise construct  $N_3(t)$  using the above algorithm with  $N_1(t)$  replaced by  $N_2(t)$ . Now, by the fundamental theorem of calculus, if  $N_j(t)$  is monotone on an interval  $J_0$ ,

$$\int_{J_0} \frac{|N'_j(t)|}{N_j(t)^5} dt \leq (\inf_{t \in J_0} N_j(t))^{-4}. \quad (6.36)$$

Therefore, by induction, (6.36), the smoothing algorithm, and the fundamental theorem of calculus,

$$\int_I \frac{|N'_m(t)|}{N_m(t)^5} dt \leq 2^{-4m+4} \int_I \frac{|N'_1(t)|}{N_1(t)^5} dt + 2. \quad (6.37)$$

Observe also that

$$N_1(t) \leq N_m(t) \leq 2^{m-1} N_1(t). \quad (6.38)$$

Next, observe that by the definition of  $N_0(t)$ ,  $N_1(t) \sim N_0(t)$ , Hölder's inequality, and the fact that as in [17], either  $N_m(t) = N_0(t)$  or  $N'_m(t) = 0$ ,

$$\begin{aligned} & \int_I \int_{|x-y| \leq \frac{R}{N_m(t)}} |u(t, y)|^2 \frac{|x-y|^2 |N'_m(t)|}{R} |\nabla u(t, x)| |u(t, x)| dx dy dt \\ & \lesssim R^3 \int_I \frac{|N'_m(t)|}{N_m(t)^4} \|u_h(t)\|_{L_x^3(\mathbf{R}^4)}^3 \|\nabla u\|_{L_x^2(\mathbf{R}^4)} dt + \int_I \frac{R^5 |N'_m(t)|}{N_m(t)^6} \|u_l(t)\|_{L_x^6(\mathbf{R}^4)}^3 \|\nabla u(t)\|_{L_x^2(\mathbf{R}^4)} dt \\ & \lesssim \int_I \frac{|N'_m(t)|}{N_m(t)^5} R^3 dt \lesssim 2^{-4m+4} K R^3 + R^3 + K^{1/2} R^5. \end{aligned} \quad (6.39)$$

Choose  $m$  so that  $2^{-4m} = R^{-3}$ . Then by (6.31),

$$\ln(R_0) \int_I \int \int_{|x-y| \leq \frac{R_0^{11/12}}{N_m(t)}} |u(t, y)|^2 |u(t, x)|^4 dx dy dt \lesssim K \ln(R_0) + K^{1/2} R_0^2 + R_0^4 + K + R_0^3 + K^{1/2} R_0^5. \quad (6.40)$$

Therefore, if  $R_0 \leq K^{1/10}$ ,

$$\int_I \int \int_{|x-y| \leq \frac{R_0^{11/12}}{N_m(t)}} |u(t, y)|^2 |u(t, x)|^4 dx dy dt \lesssim K. \quad (6.41)$$

Now since  $N_m(t) \leq 2^{m-1} N(t) \leq R_0^{3/4} N(t)$ ,

$$\int_I \int \int_{|x-y| \leq \frac{R_0^{1/6}}{N_1(t)}} |u(t, y)|^2 |u(t, x)|^4 dx dy dt \lesssim K. \quad (6.42)$$

Now, by the inverse Sobolev embedding, (5.50),  $N_1(t) \lesssim N(t)$ , (6.42) implies

$$\int_{1 \leq R \leq R_0^{1/6}} \frac{1}{R} \int_I \int \int_{|x-y| \leq \frac{R}{N_1(t)}} \int_{|y-t(t)| \leq \frac{R-C(\eta)}{N_1(t)}} \frac{N_1(t) |x-y|}{R} |\nabla u(t, x)|^2 |u(t, y)|^2 dx dy dt dR \lesssim K. \quad (6.43)$$

Also by (6.22),

$$\int_{1 \leq R \leq R_0^{1/6}} \frac{1}{R} \int_I \int_{|x-y| \leq \frac{R}{N_1(t)}} \int_{|y-y(t)| > \frac{R+C(\eta)}{N_1(t)}} \frac{N_1(t)|x-y|}{R} |\nabla u(t, x)|^2 |u(t, y)|^2 dx dy dt dR \lesssim \eta K \ln(R_0). \quad (6.44)$$

Finally, by (6.22),

$$\begin{aligned} \int_{1 \leq R \leq R_0^{1/6}} \frac{1}{R} \int_I \int_{|x-y| \leq \frac{R}{N_1(t)}} \int_{\frac{R-C(\eta)}{N_1(t)} \leq |y-y(t)| \leq \frac{R+C(\eta)}{N_1(t)}} \frac{N_1(t)|x-y|}{R} |\nabla u(t, x)|^2 |u(t, y)|^2 dx dy dt dR \\ \lesssim \int_I \int_{|x-y| \leq \frac{R_0^{1/6}}{N_1(t)}} \frac{C(\eta)}{C(\eta) + N_1(t)|x-y|} |\nabla u(t, x)|^2 |u(t, y)|^2 dx dy dt \lesssim \ln(C(\eta)) K. \end{aligned} \quad (6.45)$$

Then by taking  $m$  such that  $2^{4m} = R_0^{5/9}$ , by (6.24) - (6.29), (6.43) - (6.45), since  $N_m(t) \geq N_1(t)$ ,

$$\begin{aligned} R_0^{2/3} &\gtrsim \int_{1 \leq R \leq \frac{R_0^{1/6}}{N_m(t)}} \frac{1}{R} \int_I \frac{d}{dt} M_R(t) dt dR \\ &\gtrsim \ln(R_0) \int_I \int_{|x-y| \leq \frac{R_0^{11/72}}{N_m(t)}} |u(t, y)|^2 |u(t, x)|^4 dx dy dt dR \\ &\quad - \ln(C(\eta)) K - \eta K \ln(R_0) - R_0^{1/2} - R_0^{-1/18} K - K^{1/2} R_0^{5/6}. \end{aligned} \quad (6.46)$$

Also since  $N_m(t) \leq 2^{5/36} N_1(t)$ ,

$$\begin{aligned} \int_I \int_{|x-y| \leq \frac{R_0^{1/72}}{N_1(t)}} |u(t, y)|^2 |u(t, x)|^4 dx dy dt \\ \lesssim \eta K + \frac{C(\eta)}{\ln(R_0)} K + \frac{o(\ln(R_0))}{\ln(R_0)} K + R_0^{-1/18} K + K^{1/2} R_0^{5/6} + R_0^{1/2} + R_0^{2/3}. \end{aligned} \quad (6.47)$$

Therefore, since  $N_1(t) \leq N(t)$  there exists a sequence  $t_n \in \mathbf{R}$ ,  $R_n \nearrow \infty$  such that either

$$N(t_n)^2 \int_{|x-x(t)| \leq \frac{R_n}{N(t)}} |u(t, x)|^2 dx \rightarrow 0, \quad (6.48)$$

or

$$\int_{|x-x(t)| \leq \frac{R_n}{N(t)}} |u(t, x)|^4 dx \rightarrow 0. \quad (6.49)$$

However, if  $u$  lies in a precompact set modulo scaling and translation symmetries,  $\|u(t)\|_{L_x^4(\mathbf{R}^4)} \gtrsim \|u(t)\|_{\dot{H}^1(\mathbf{R}^4)}$ . Therefore, (6.48) or (6.49) imply that  $u \equiv 0$ .  $\square$

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